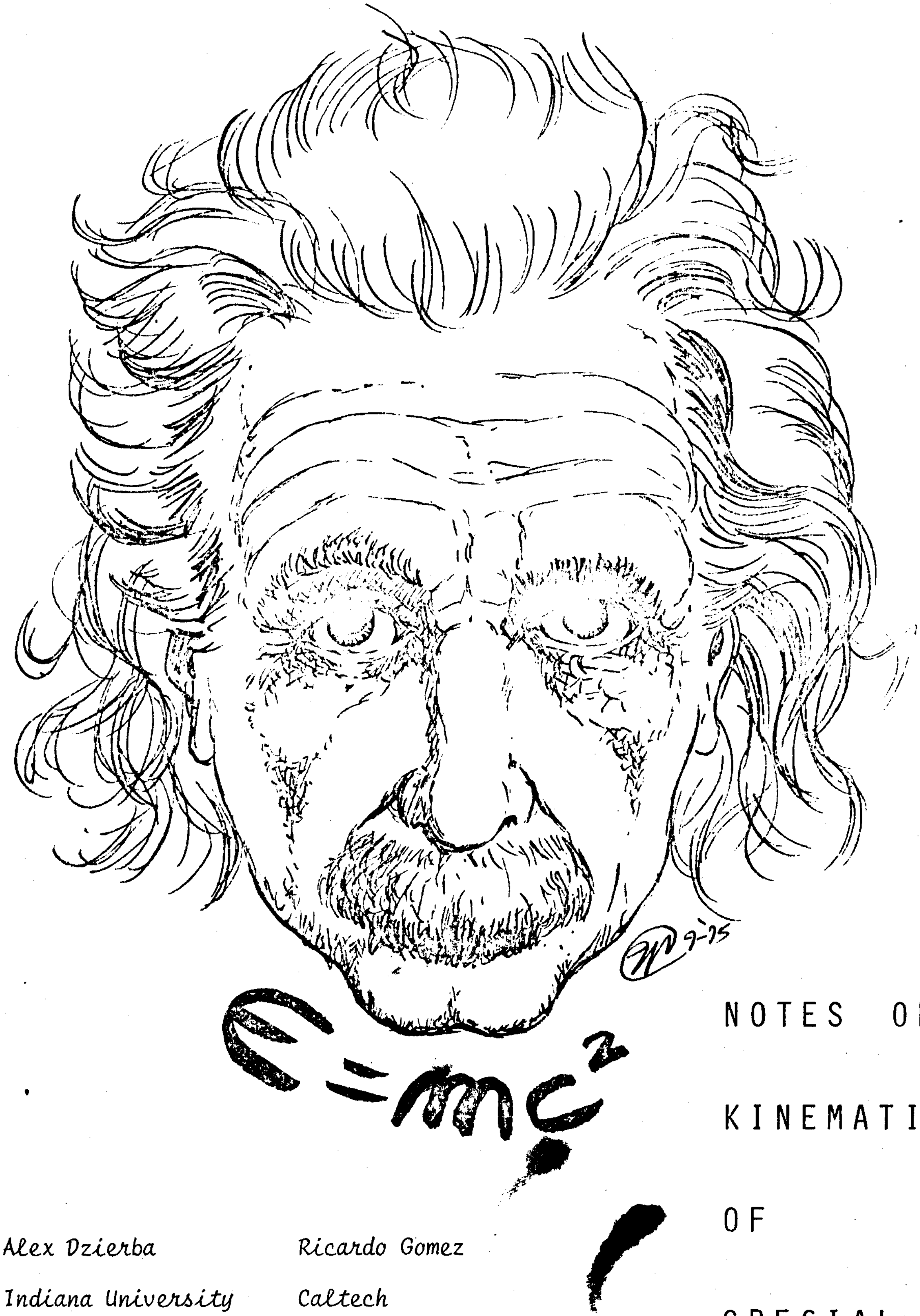

NOTES ON THE KINEMATICS OF SPECIAL RELATIVITY

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Contents:

(I) INTRODUCTION

These notes discuss how the predictions which follow from the theory of special relativity, such as time dilation and length contraction, can be derived quite easily from a simply stated axiom. That axiom is:

ALL THE LAWS OF PHYSICS ARE THE SAME IN EVERY INERTIAL REFERENCE FRAME.

Axiom of special relativity

A careful application of the above axiom will lead us to what we are after here, the Lorentz transformation, of which time dilation and length contraction are a part. What the Lorentz transformation does for us is link measurements made in one reference system with measurements made in another reference system moving with constant velocity with respect to the first. Once we have derived the Lorentz transformation and have built up a good comfortable feeling with it, we will study the consequences of applying the Lorentz transformation.

No matter what experiment we do we cannot tell whether or not we're moving at a constant velocity.

The precise genius of Einstein's work in inventing the theory of special relativity was in making clear what we mean by certain basic notions about which we have strong prejudices. These include the notions of time, measurements and reference frames within which measurements of time and distances, that is space-time measurements, are made. It is important to define and understand these notions very clearly. Once one does, the "common-sense-defying" predictions of special relativity follow in a way that is amazingly simple from the axiom stated above.

(II) THE GALILEAN TRANSFORMATION

Before discussing the Galilean transformation, we will define a very important concept, that of the inertial reference frame, or simply the inertial frame.

Definition of inertial frame

AN INERTIAL FRAME IS A FRAME OF REFERENCE IN WHICH A BODY NOT UNDER THE INFLUENCE OR FORCES AND INITIALLY AT REST WILL REMAIN AT REST.

Notice that with this definition, accelerating frames, such as rotating frames, are not considered to be inertial frames.

Now the question is, how does one link measurements made in two inertial frames moving with constant velocity with respect to one another. To consider a specific example, let's take reference frame 0 moving with velocity v with respect to

Galilean transformations

reference frame O' along the x axis. (See Fig. 1) The transformation from one reference frame to the other of space and time coordinates is almost "too obvious" to require stating it:

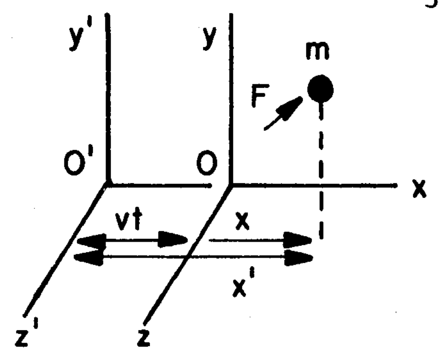


Figure 1

$$\begin{aligned} x' &= x + vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \qquad (1)$$

Note that equations (1) imply that the two origins (0 and 0') coincide at t=0.

The above set of equations is referred to as the Galilean transformation and seems almost obvious, or is it?

Let us consider what implications this Galilean transformation has when we analyze physics with Newton's second law:

$$\vec{F} = m\vec{a} \qquad (2)$$

If we have a particle of mass m as shown in Fig. 1, the 3 component equations of Eq. (2) as given in frame O are:

$$\begin{aligned} F_x &= m\frac{d^2x}{dt^2} \\ F_y &= m\frac{d^2y}{dt^2} \\ F_z &= m\frac{d^2z}{dt^2} \end{aligned} \qquad (2')$$

The Galilean transformation leaves the form of the law:

F = ma

unchanged.

These equations then tell us what the motion of the particle will be in frame O. If we transform to frame O' using Eq. (1), we note that $t = t'$ and $d^2x'/dt'^2 = d^2x/dt^2$, the latter following by differentiating $x'=x + vt$ with respect to $\underline{t}$ twice. The second derivatives with respect to time of y' and z' are equal to those of y and z, respectively. Also the components of the force, F, are the same along x and x'. Similarly, this holds true for the components along the y and z directions. The result: Newton's laws are the same in any inertial frame. This we learn from applying the Galilean transformation.

According to Newton's laws and the Galilean transformation it is impossible to detect whether a frame of reference is at rest or moving at a constant velocity. Only relative motion can be detected. This is the principle of relativity.

The other physical phenomena in classical physics are electromagnetic. All these are successfully described by the four Maxwell's Equations which predict the existence of electromagnetic waves; light is just that, an electromagnetic wave. It was found that when one applied the Galilean transformations to Maxwell's equations, the mathematical form of the equations changed. Moreover, the electromagnetic waves, whose existence was predicted by Maxwell's equations, had a propagation velocity which was different in inertial frames which were moving with uniform velocity with respect to one another. There was another way of looking

On the other hand, according to Maxwell's equations and the Galilean transformation absolute uniform motion can be detected. Therefore, taken together they violate the principle of relativity.

at this problem. All waves travel in a medium; sound in air, for example. The velocity of the wave will change if the observer is moving relative to the medium. It was thought that there had to be a medium for the electromagnetic wave, light. This medium was called the ether. One could imagine that the ether is fixed in some frame. It would follow then that electromagnetic disturbances, in other words, light, would travel with different velocities in different moving inertial frames. This search for the ether was carried out in an experiment in 1887 by Michelson and Morley. Their result was a null result. There was no ether or, to put it another way, experimentally it was observed that the speed of light, c, was the same in moving inertial frames.

Michelson and Morley could not detect absolute motion using electromagnetic effects (see problem 1). What was wrong?

the Galilean transformation ?
Maxwell's equations (electromagnetism) ?
Newton's laws ?
Principle of Relativity ?

Einstein found that Maxwell's and the Principle of Relativity were correct, and that the Galilean transformation (which became the Lorentz transformation) and Newton's second law had to be modified.

So the situation before Einstein and the theory of special relativity was the following:

The laws of classical mechanics (Newton's laws) are the same, that is, have the same mathematical form, in all inertial frames. This we see to be consistent with the Galilean transformation. However, this is not the situation with electromagnetic laws, Maxwell's equations. However, although we might expect an ether to exist which would be the medium for electromagnetic waves, the experiment of Michelson and Morley showed that this ether does not exist. Consequently the speed of light in vacuum is independent of the motion of the observer and independent of the motion of the source.

THE IMPORTANT ASSUMPTION IN EINSTEIN'S THEORY OF SPECIAL RELATIVITY

(III) SOME BASIC NOTIONS

Before we go on to deriving the Lorentz transformation, we define here some very basic notions. It is important to do this, since it was only after earlier prejudices about these notions were discarded that the dilemma of classical physics was cleared up and special relativity invented.

a. Event: The concept of an event in physics is just as fundamental as the concept of place is in surveying. An event is specified by a place and time of happening. Therefore, one needs four coordinates to describe an event: (x, y, z, t).

Event

Example of an event: a flash of light

b. Time: This is most crucial. Time, at least as far as it concerns the physicist, is not a metaphysical, absolute "flowing" thing. Rather, it is no more sacred than space. If we have

Time

two events, each specified by space-time coordinates (x, y, z, t) and (x_o, y_o, z_o, t_o), we measure the distance between these events in meters, for example. That means we hold a meter stick in our hand and count how many tick marks on the meter stick lie between x and x_o. We do the same for y and y_o and z and z_o. What about t and t_o.

Note: Wheeler and Taylor in their Spacetime Physics (pg. 1 ff) tell an amusing parable which points out time-space prejudices.

Well, t and t_o should be treated exactly the same way as x and x_o or y and y_o and z and z_o. Only this time instead of counting how many ticks on the meter stick are between x and x_o, we measure how many ticks of a clock there are between t and t_o. So we measure the x, y and z separations between events in meters, but the time separations in seconds. With the discovery that light as determined by all observers always travels at the same speed, c, we can use such a speed as a conversion factor between time intervals and space intervals.

c. Clocks: Clocks are things which produce ticks which allow us to measure time intervals.

Clocks

Examples are: 1) mechanical clock

2) rotating earth

3) your own pulse

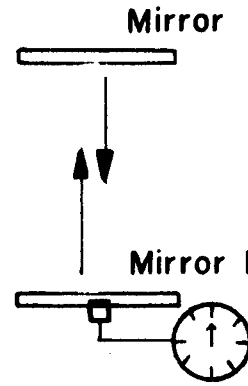


Fig. 2: A clock

We'll take as our clock, in what follows, the following device: Let's have two mirrors as shown in Fig. 2 with a light pulse bouncing between them. Every time a light pulse is detected at M we have a tick. If M and N are ½ cm apart, each tick corresponds to 1 cm of light travel time. Every time we detect a flash at M, we have a needle on a dial advance one tick.

The period of any other clock (e.g. a wristwatch) will be a multiple of the period of our special clock. If we now take both clocks into a moving inertial frame, the ratio of the periods measured in this frame must remain unchanged, otherwise we could determine that we are moving.

(IV) THE PRINCIPLE OF SPECIAL RELATIVITY

We can summarize the principle of special relativity in the following two statements:

ALL THE LAWS OF PHYSICS ARE THE SAME IN EVERY INERTIAL FRAME
and
THE SPEED OF LIGHT IN VACUUM IS THE SAME IN ALL INERTIAL FRAMES

What the first statement says is that both the mathematical form and numerical values of the physical constants in the laws of physics are the same in all inertial frames. Thus the laws of physics cannot be used in anyway to determine the absolute velocity of a particular inertial frame; only relative velocities can be determined.

Remember:

You cannot do any experiment to indicate whether or not you are moving.

The second statement says the speed of light measured in any frame moving with constant velocity always has the same value. This of course is the empirical observation made by Michelson and Morley.

(V) THE LORENTZ TRANSFORMATION

Lorentz transformation

In Section I, we wrote down the Galilean transformation which related measurements made in two inertial frames. However, problems arose with applying the Galilean transformation to electromagnetic phenomena. Let's start over, taking care now to apply our basic notions of time correctly. We'll consider the transformation between the two frames as shown in Fig. 3. Before writing the general transformation equations, we consider three classes of measurements:

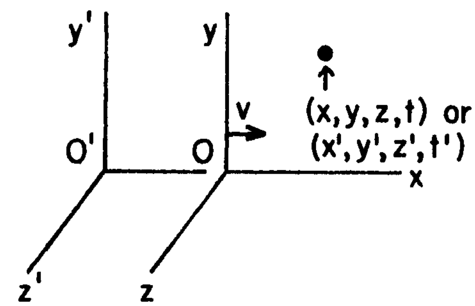


Fig. 3

a. measurements of space intervals perpendicular to the direction of the velocity v (transverse dimensions);

b. measurements of time intervals;

c. measurements of space intervals along the direction of motion.

Notice the language we're using. We are always after a relation (the Lorentz transformation) describing <u>intervals between two events</u> as seen in two inertial frames.

Transverse Dimensions

<u>(a) Transverse dimensions</u>: Assume you have the two systems moving relative to each other along

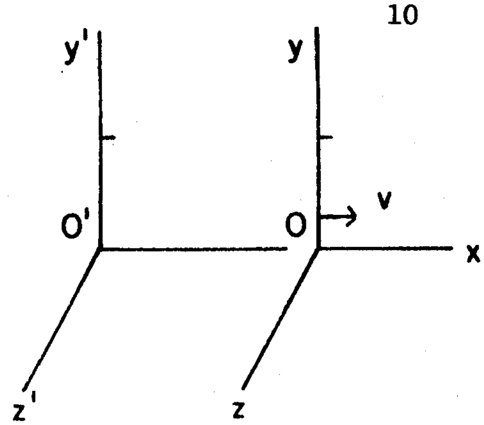


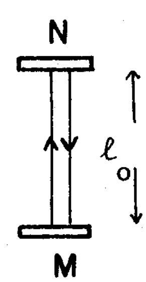
Fig. 4

the x (x') axis. Let the observer in O make a mark in O' on the y'x' plane at height y (Fig. 4). Similarly let the observer in O' make a mark in O at a height y and assume they both agreed before hand to mark at the same height in inches. The two marks had better coincide, because otherwise there would be a way of determining which of the two inertial frames is moving. This, of course, would violate the principle of special ralativity. If you think a bit, you come to the conclusion that intervals measured along y in frame O agree with measurements of intervals along y in frame O'. Similarly for z. The result: <u>Transverse intervals are unchanged under a Lorentz transformation</u>.

Transverse intervals are unchanged when going from one inertial frame to another.

<u>(b) Measurements of time intervals</u>: Suppose we have a clock which is at rest in the frame O. The clock is of the type described in Fig. 2. Every time the light pulse reaches M, we call that a clock tick. To be more precise, we say that in frame O we have two events. Event 1: light leaves mirror M. Event 2: light arrives at mirror M. Notice that these two events occur at the same place (in space) in frame O but at different times. We call the time interval between these two events Δt. What is the time interval between these two

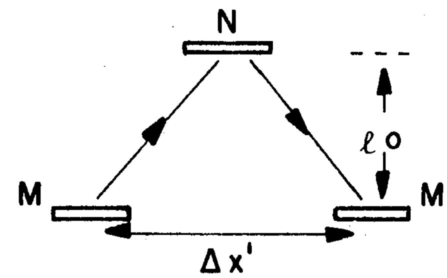
Measurement of time Intervals



(a) Clock is at rest in frame O as viewed from observer in frame O.

Fig. 5

events, $\Delta t'$, as observed in frame $0'$? In order to answer this, look at Fig. 5(b). We notice that since 0 is moving relative to $0'$, an observer in $0'$ does not see the two events (1: light leaves M; 2: light arrives at M) occur in the same place in $0'$. The reason is clear. While the light travels from M to N and back to M, the mirror M has moved in the $0'$ frame. As a matter of fact, seen from $0'$ the light travels the path shown in Fig. 5(b). As seen in $0'$, the mirror M has moved $\Delta x'$. The time interval between the two events as seen in $0'$ is $\Delta t'$. Now



(b) The same clock in frame 0' as viewed by an observer at rest in 0'

Fig. 5

$$\Delta x' = v\Delta t'$$

The path length the light travels in $0'$ is

$$2\left[\ell_o^{\,2} + (v\Delta t'/2)^2\right]^{1/2}$$

Notice that since ℓ_o is a transverse interval, it is the same for both 0 and $0'$. But since c has the same numerical value in all inertial frames then

Time Dilation

$$2\left[\ell_o^{\,2} + (v\Delta t'/2)^2\right]^{1/2} = c\Delta t'$$

or

$$\Delta t' = \frac{2\ell_o/c}{\sqrt{1-v^2/c^2}} \qquad (3)$$

Notice, however, that $2\ell_o/c$ is just Δt. We'll now introduce some commonly used notation:

$$\beta = v/c \tag{4}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \tag{5}$$

with this we write the general result:

$$\boxed{\Delta t' = \gamma \Delta t \quad (\text{for } \Delta x = 0)} \tag{6}$$

Since $\gamma > 1$, we see that moving clocks appear to run slower.

There once was a lass named
Miss Bright
Who could travel much
faster than light
She departed one day
In an Einsteinean way
And came back on the
previous night.
- G. Gamov

This is not just mathematical trickery. Time dilation is an empirically observed fact. The classic example is the muon decay. One refers to the lifetime of a particle as the time it takes the particle to decay if it's produced at rest. This number is known for muons. Now one can then use this lifetime to predict how far a muon of a certain momentum ought to travel after production before decaying without using time dilation. The predicted distance of travel will always be shorter than what is observed by exactly the right amount predicted from time dilation.

For a discussion of the time dilation experiment using muons, see French, Special Relativity, pp 97 - 104.

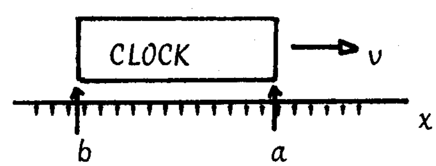
"Time dilation works"

Question: In reference to the above paragraph, what are the Δt, $\Delta t'$ and Δx in Eq. (6)? What are the two events? Be precise in your answer.

(c) Measurements of space intervals along the direction of motion: In order to see how space intervals along the direction of motion transform

Measurement of space intervals along direction of motion let's once again fall back on our trusty clock. Consider two identical clocks and orient them as shown in Fig. 6(a). If a light pulse leaves mirror M and travels to N and N' the reflected pulses will arrive at M simultaneously. If we now observe the clocks from a frame moving along the x-direction the reflected pulses will still arrive at M simultaneously. Therefore the observed periods of both clocks are the same. In the rest frame of the clocks the period is $2\ell_o/c$. In the moving frame the observed period is dilated by the factor γ as in equation (6).

Now let's concentrate on the clock in Fig. 6(b). It's important to define precisely what we mean by the length of our clock (or any object for that matter) in a moving frame. To measure the length of an object when it is moving in relation to us, what we must do is to measure the coordinates of the two ends at the same time. This we can do, for instance, by making marks on a scale when the clock goes by, as shown in the figure. We will call events a and b the marking of the two ends on the scale. Since these markings must be made at the same time $t'_a - t'_b = 0$, and $x'_a - x'_b = \ell$.



We measure the length of our clock in a moving frame by simultaneously marking the points on the x'-axis which coincide with the two ends of the clock.

As observed in the moving frame (Fig. 6(c) the length of the clock is ℓ where ℓ is not necessarily equal to ℓ_o. We will now consider another pair of events. Event d is the light pulse leaving mirror M and event e is the light pulse returning to mirror M. The time interval between these two events is the period observed for the clock in the moving frame, i.e:

$$\text{Period} = t_e - t_d = \gamma(2\ell_o/c).$$

This period is also equal to the time it takes light to travel from mirror M to mirror N and back again to mirror M, taking into account the fact that the mirrors are moving. The time interval for the trip from M to N is $\Delta t_1'$ where

$$\ell + v\Delta t_1' = c\Delta t_1'$$

and the time interval for the return trip is $\Delta t_2'$ where

$$\ell - v\Delta t_2' = c\Delta t_2' \ .$$

Now:

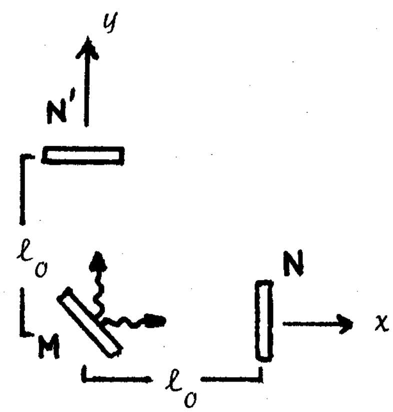
$$t_e - t_d = \Delta t_1' + \Delta t_2'$$

$$= \frac{\ell}{c - v} + \frac{\ell}{c + v}$$

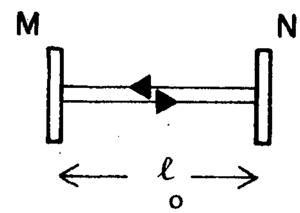
$$= \frac{2\ell c}{c^2 - v^2} = \frac{2\ell/c}{1 - v^2/c^2} = \gamma^2(2\ell/c)$$

But we also have

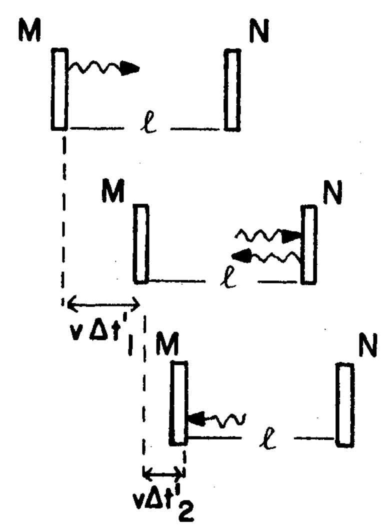
$$t_e - t_d = \text{period} = \gamma(2\ell_o/c)$$



(a) Two clocks oriented at 90°



(b) Clock as seen from frame in which clock is at rest



(c) Same clock as seen from frame moving with velocity v along ℓ_o

Fig. 6

Therefore

$$\ell = \ell_o/\gamma .$$

Let's return to the previous pair of events (a and b) which we used to measure the clock length ℓ in the moving frame. There we had

$$x'_a - x'_b = \Delta x' = \ell = \ell_o/\gamma$$

Length Contraction

$$t'_a - t'_b = \Delta t' = 0$$

and

$$x_a - x_b = \Delta x = \ell_o$$

We now know how to relate the space intervals between two events which occur at the same time in one of the frames. This relation is

$$\boxed{\Delta x' = \frac{\Delta x}{\gamma} \quad (\Delta t' = 0)} \qquad (7)$$

This phenomenon is called length contraction.

There was a young lad
named Fisk
Whose fencing was ex-
ceedingly brisk
So fast was his action
the Fitzgerald contraction
Reduced his rapier
to a disk.

- G. Gamov

(d) Simultaneity

Simultaneity (or the lack of it)

Before we go on to deriving the general form of the Lorentz transformation we will discuss the implications of results we have just obtained regarding the concept of simultaneous events. In Fig. 7 a light source, L, is located midway between two detectors D_1 and D_2. At t=0 a flash of light is produced by the light source and is detected simultaneously at $t=\ell_o/c$ by D_1 and D_2. Let's refer to the arrival of light at D_1 as event 1. The arrival of light at D_2 as event 2. The time interval between these events as observed in the frame in which L, D_1 and D_2 are at rest is Δt where

$$\Delta t = t_2 - t_1 = 0$$

We say that the two events are simultaneous.

Simultaneous Events

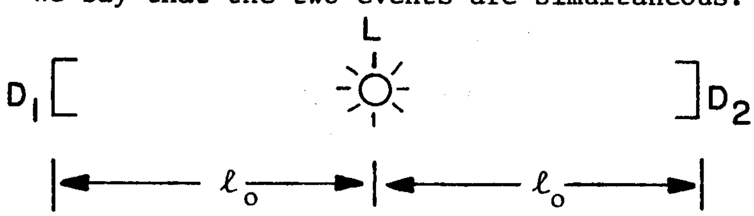


Fig. 7

Simultaneous events in one frame viewed from another frame

Suppose now that we observe these same two events (light arriving at D_1 and light arriving at D_2) from a frame 0' moving along the -x direction with speed v with respect to frame 0. Fig. 8 shows what is observed in the 0' frame. At t' = 0, the light source flashes

Figure 8a). Of course the distance between D_1 and the light source and D_2 and the light source is now observed to be Lorentz-contracted by the factor $1/\gamma$. At $t' = t_1'$ event 1 occurs: light is detected by D_1 (Fig. 8b). In this case the detector is moving forward at speed v to meet the light moving at velocity c. At $t' = t_2'$ event 2 occurs: light is detected by D_2 (Fig. 8c). However now D_2 was moving at speed v away from the light.

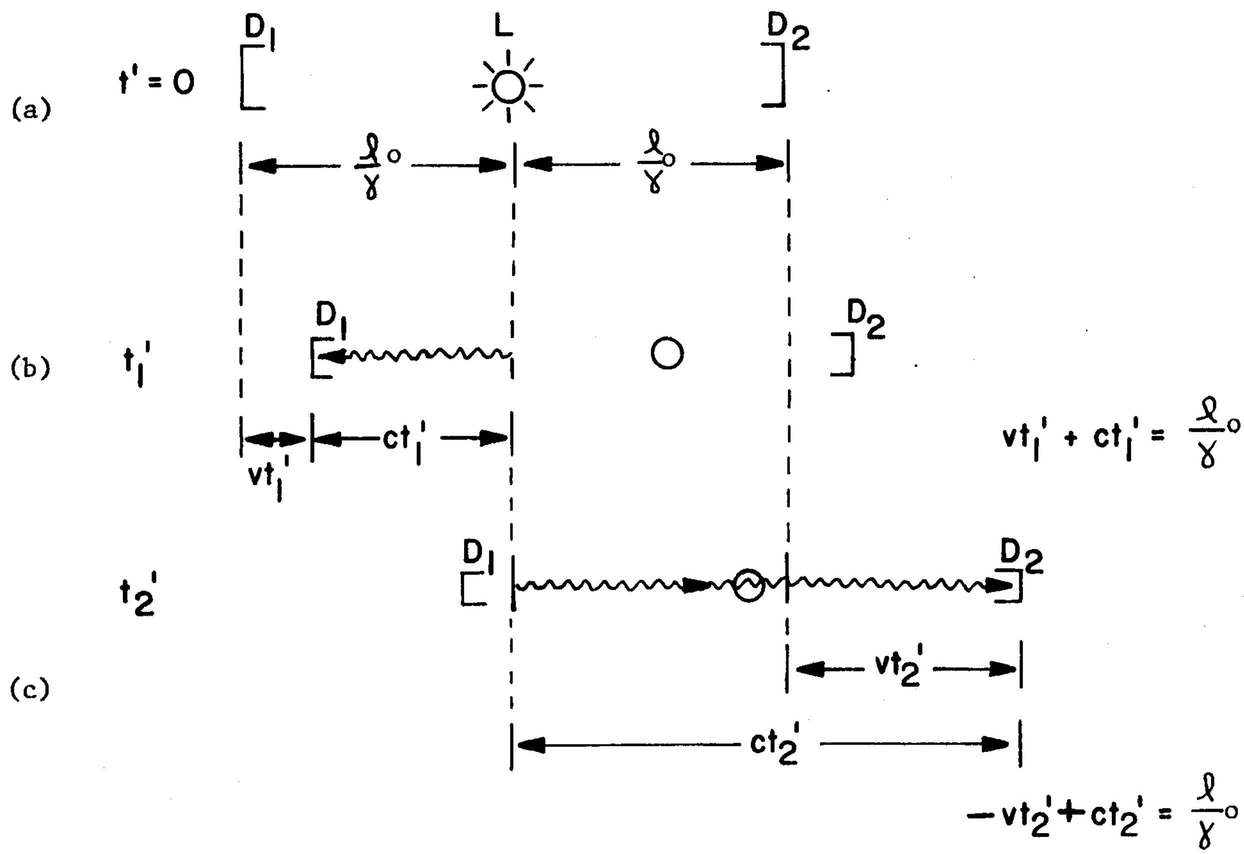


Fig. 8

From Fig. 8b and Fig. 8c we see that

$$vt_1' + ct_1' = \frac{\ell_o}{\gamma}$$

and

$$-vt_2' + ct_2' = \frac{\ell_o}{\gamma}$$

Solving for t_1' and t_2'

$$t_1' = \frac{\ell_o}{\gamma}\,\frac{1}{c+v}$$

$$t_2' = \frac{\ell_o}{\gamma}\,\frac{1}{c-v}$$

so

$$\Delta t' = t_2' - t_1' = \frac{2\ell_o}{\gamma}\,\frac{v}{c^2 - v^2}$$

or

$$\Delta t' = 2\beta\gamma\,\frac{\ell_o}{c} \qquad (8)$$

So the result is that $\Delta t' \neq o$ and consequently events 1 and 2 are not simultaneous for an observer in the moving frame. So two events simultaneous but spatially separated in a given frame are not simultaneous in another frame. What led to this lack of simultaneity? The invariance of the speed of light, c, is the answer.

(e) The general form of the Lorentz Transformation: We now have come to the problem of relating two events which are separated in both space and time. We then observe these intervals in another frame moving with respect to the original frame. So far we have con-

General form of the Lorentz transformation sidered the transformation of time intervals between events occurring at the same point in space (that led to time dilation) and space intervals between events occurring at the same time (which led to length contraction). We're now ready to tackle a more difficult problem.

Let us now orient our clock in its rest frame so that it is neither aligned along the x- or y-axis (see Fig 9(a)). A light pulse travels from mirror M to N. This gives us two events separated in space and time. The two events are: Event 1: light leaves mirror M; Event 2: light arrives at mirror N. In the rest frame the x and y intervals between these events are ℓ and d respectively. Let us now go to a frame moving along the x direction and ask what the space-time intervals between the same two events are in this frame. In the rest frame:

$$\Delta x = \ell \tag{9}$$

$$c\Delta t = \sqrt{d^2 + \ell^2} \tag{10}$$

While in the moving frame:

$$\Delta x' = \frac{\ell}{\gamma} + v\Delta t' \tag{11}$$

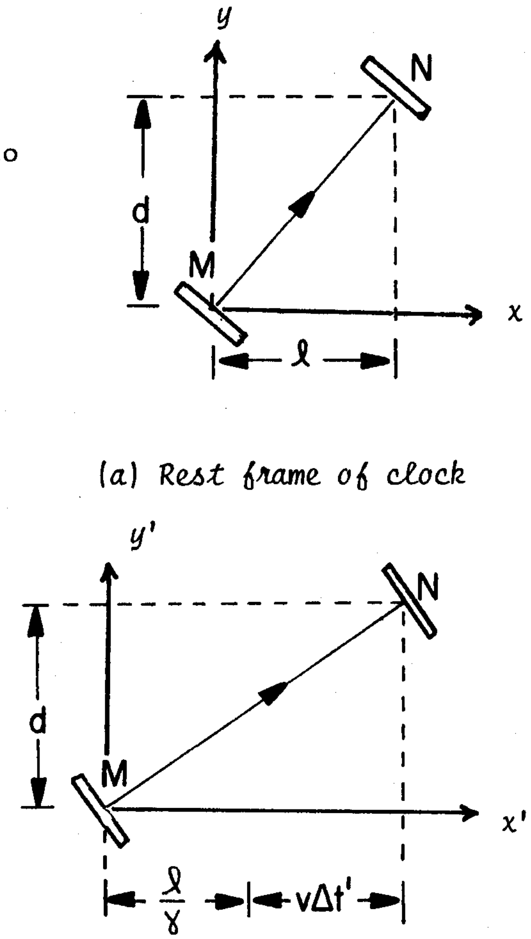
$$c\Delta t' = \sqrt{d^2 + \Delta x'^2} \tag{12}$$

From (9) and (11):

$$\Delta x' = \frac{\Delta x}{\gamma} + v\,\Delta t'$$

or

$$\Delta x = \gamma(\Delta x' - v\Delta t') \tag{13}$$



(a) *Rest frame of clock*

(b) *Same clock as seen from a frame moving along the x-axis*

Fig. 9

Of course the transverse space intervals, Δy and Δz are unchanged in going from the rest frame to the moving frame.

From (10), (12) and (13) and a little algebra (see problem 3.)

$$\Delta t = \gamma(\Delta t' - \frac{v}{c^2}\,\Delta x') \qquad (14)$$

There we have it, the complete Lorentz transformation equations. Adding the result that transverse intervals do not change we have:

$$\boxed{\begin{aligned} \Delta x &= \gamma(\Delta x' - v\Delta t') \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \\ \Delta t &= \gamma(\Delta t' - \frac{v}{c^2}\,\Delta x') \end{aligned}} \qquad (15)$$

General Form of the Lorentz Transformation

The primed intervals are measured in the 0' frame which is moving to the left in the 0 frame with velocity v. To get the inverse relations simply interchange primed and unprimed intervals and replace v by -v. (see problem 4)

How general are equations 15 ?

The two events which we used to arrive at equations (15) are a special pair of events. These are two events which are <u>connected by a light pulse</u> so that in any given frame, the space interval, Δs, between the two events is equal to the time interval, Δt, between the two events times c, the speed of light. What if we have a pair of events which cannot be connected by a light pulse ? Can the space-time intervals between these events be transformed from one inertial frame to the other by using equations (15) ? The answer is <u>yes</u>.

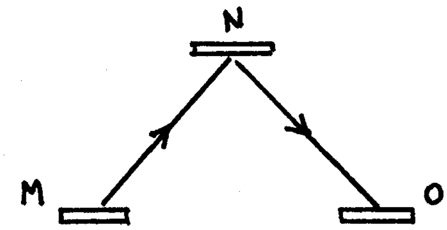
We will now show why equations 15 are more general than they might first appear.

One possible pair of events not connected by a light pulse is a pair for which $c\Delta t > \Delta s$ (in the language of the physicist these events are said to be separated by time-like intervals). Consider the figure to the right which shows two of our clocks. A light pulse travels from mirror M to N to O. We have 3 events:

event 1: light pulse leaves mirror M

event 2: light pulse reflects off mirror N

event 3: light pulse arrives at mirror O



Arrangement of two clocks showing example of two events which cannot be connected by a light pulse.

Events 1 and 2 are connected by a light pulse as are events 2 and 3. However events 1 and 3 cannot be connected by a light pulse since $c\Delta t > \Delta s$. Since equations (15) can be used to transform space-time intervals (t_2-t_1 and s_2-s_1) between events 1 and 2 as well as space-time intervals (t_3-t_2 and s_3-s_2) between events 2 and 3 it should be obvious that they therefore allow us to transform the space-time intervals (t_3-t_1 and s_3-s_1) between events 1 and 3.

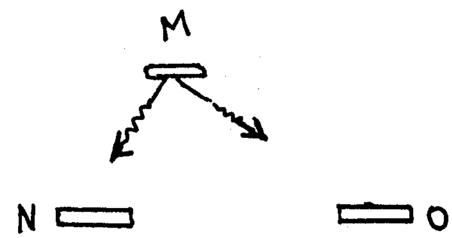
Another possibility is to have two events for which $\Delta s > c\Delta t$ (such events are said to be separated by a space-like interval). The figure on the next page shows another arrangement of clocks in which we have a light pulse leaving mirror M and arriving at mirrors N and O. Note that the two clocks have differing lengths. Again we have 3 events:

event 1: light pulse leave mirror M

event 2: light pulse arrives at mirror N

event 3: light pulse arrives at mirror O.

As before, events 1 and 2 are connected by a light pulse as are events 1 and 3. But now events 2 and 3 cannot be connected by a light pulse. In this case $\Delta s > c\Delta t$. Since equations (15) can be used to transform intervals between events 1 and 2 and to transform intervals between 1 and 3, they can also be used to transform intervals between events 2 and 3. So we see that equations (15) can be used to transform space-time intervals between any pairs of events from one inertial frame to another.



Another arrangement of clocks to show an example of two events which cannot be connected by a light pulse.

(VI) CAUSALITY

One often hears the statement that information cannot be transmitted with speeds greater than the speed of light. In fact there has been a lot of discussion in physics literature lately about the possibility of the existence of tachyons (faster than light particles). In order to study the existence of tachyons within the theory of relativity, consider the following thought experiment. Suppose in a fit of passion, Uxl of planet X decides to do away with his friend, Nork. He takes aim at him with a special ray-gun (which will only stun him, please) whose rays travel with a velocity V greater than the speed of light. In their frame we have two events: Uxl firing his ray gun and Nork dropping upon receiving the stunning ray. So in the Uxl-Nork frame, the two events are separated by Δx and Δt.

Tachyons

Now let's look at these two events from another frame which is moving with velocity v with respect to Uxl & Nork. In this moving frame, an observer sees the two events separated by the time interval $\Delta t'$:

$$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$$

But in the Uxl-Nork frame, Δx is traveled by the powerful ray in time Δt so

$$\Delta x = V\,\Delta t$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2} V\,\Delta t\right)$$

$$= \gamma \Delta t\left(1 - \frac{v}{c}\,\frac{V}{c}\right) \qquad (16)$$

But notice, if $\frac{V}{c}$ is greater than 1, it's possible for $\frac{v}{c} \cdot \frac{V}{c}$ to be greater than one with $\frac{v}{c} < 1$ (a physically realizable situation) which means that the factor in Eq. (16) multiplying Δt could be negative. This would mean our moving observer could see Nork dropping before Uxl fires his ray gun. See Fig. 10. This violates causality. Conclusion: Causality implies that no information can travel with speed greater than c, the speed of light.

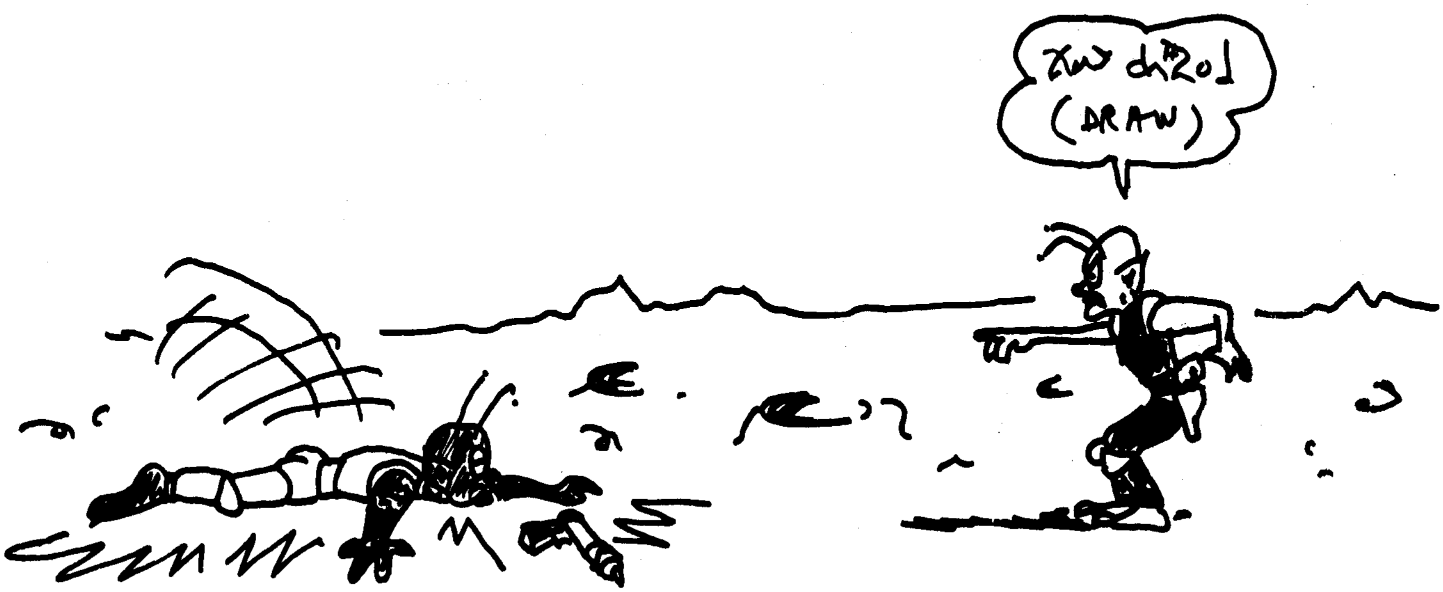


Fig. 10: A violation of causality

(VII) ADDITION OF VELOCITIES

Suppose we have a particle in frame O' moving with velocity v' in frame O'. (Fig. 11) The components are v'_x, v'_y, and v'_z. We want to find the velocity of this particle as seen from frame O which is moving with velocity V with respect to O'. Do we simply take $\vec{v}' = \vec{v} - \vec{\beta}c$? The answer is no. We explicity calculate $v_x = \Delta x/\Delta t$ and so on.

Addition of Velocities

From Eqs. (15): Note: $\beta = \frac{V}{c}$

$$\Delta x = \gamma(\Delta x' - V\Delta t')$$

$$\Delta y = \Delta y'$$

$$\Delta z = \Delta z'$$

$$\Delta t = \gamma(\Delta t' - \frac{V}{c^2}\Delta x')$$

So

$$v_x = \frac{\Delta x}{\Delta t} = \frac{\gamma(\Delta x' - V\Delta t')}{\gamma(\Delta t' - \frac{V}{c}\ \Delta x')}$$

or:

$$\boxed{v_x = \frac{v'_x - V}{1 - \frac{V}{c^2}v'_x}} \qquad (17)$$

$$V_y = \frac{\Delta y}{\Delta t} = \frac{\Delta y'}{\gamma(\Delta t' - \frac{V}{c^2}\Delta x')}$$

or:

$$\boxed{v_y = \frac{\sqrt{1-\beta^2}\ v'_y}{1 - \frac{V}{c^2}\ v'_x}} \qquad (18)$$

and

$$\boxed{v_z = \frac{\sqrt{1-\beta^2}\ v'_z}{1 - \frac{V}{c^2}\ v'_x}} \qquad (19)$$

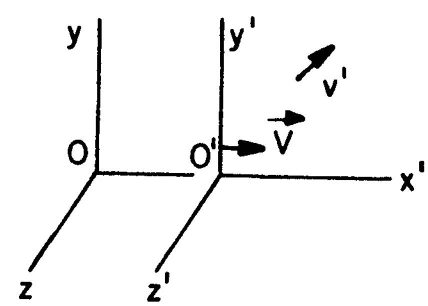


Fig. 11

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IX) Problems and Questions

1. The basic idea of the Michelson-Morley experiment was to measure the velocity of light as seen from a frame of reference fixed with respect to the earth, in two perpendicular directions. A schematic of the experiment is shown in Fig. 1. A source of light S, is collimated by lens C. M is a half silvered mirror, M_1 and M_2 are perfectly reflecting mirrors. Part of the light from S follows the path C-M-M_2-M-T; the other part of the light follows the path C-M-M_1-M-T. At the telescope, T, one looks at the light intensity which will be a function of the phase relation of the recombining beam. Assume now that the inferometer is traveling through the ether with velocity v.

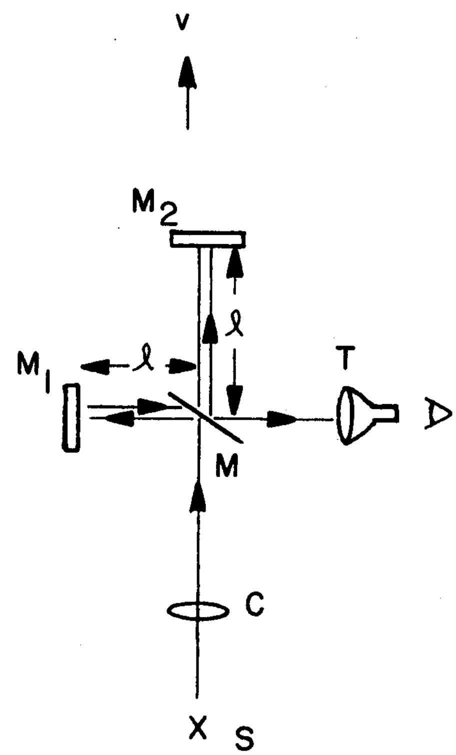

Fig. 1

Michelson-Morley Experiment

ANALYZE THIS EXPERIMENT USING NON-RELATIVISTIC PHYSICS. *That is, assume that c is the speed of light in the ether frame of reference.*

a) Show that, referring to our figure, the two combining light rays are out of phase by an amount $\Delta\phi_1$, where:

$$\Delta\phi_1 = 2\pi \frac{\ell v^2 *}{\lambda c^2}$$

(b) Rotate the appartus through 90° and make a second measurement. In that case

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*these equations are valid only if v < < c.

The change in phase is $\Delta\phi_2$ where:

$$\Delta\phi_2 = -2\pi \frac{\ell v^2}{\lambda c^2}.$$

(c) The change in phase before and after rotating the appartus is given by

$$\Delta = 4\pi \frac{\ell v^2}{\lambda c^2}$$

Michelson and Morley used $\ell = 10^3$cm and $\lambda = 6 \times 10^{-5}$ cm. If we assume that the velocity of the earth with respect to the ether is $\sim 10^{-4}c$, we get $\Delta \approx 3$ radians. With this value of Δ how would observations before and after the 90° rotation compare?

For the fun of it calculate Δ using special relativity.

2. Fill in the algebraic steps leading to equation (8).

Some healthy algebraic "grunge"

3. Fill in the algebraic steps leading to equation (14).

4. Show explicitly, by inverting equations (15) that the inverse Lorentz transformations are given by replacing v with -v.

Lorentz transformation

5. Show that if L_o^3 is the volume of a cube measured in the rest frame of the cube then the volume viewed from a frame moving with velocity v in a direction parallel to an edge of a cube is

Volume Contraction

$$L^3 = L_o^3/\gamma$$

6. Show that in the limit of an infinite value for the speed of light, c, the Lorentz transformation equations (15) reduce to the

Galilean transformatinn equations (1).

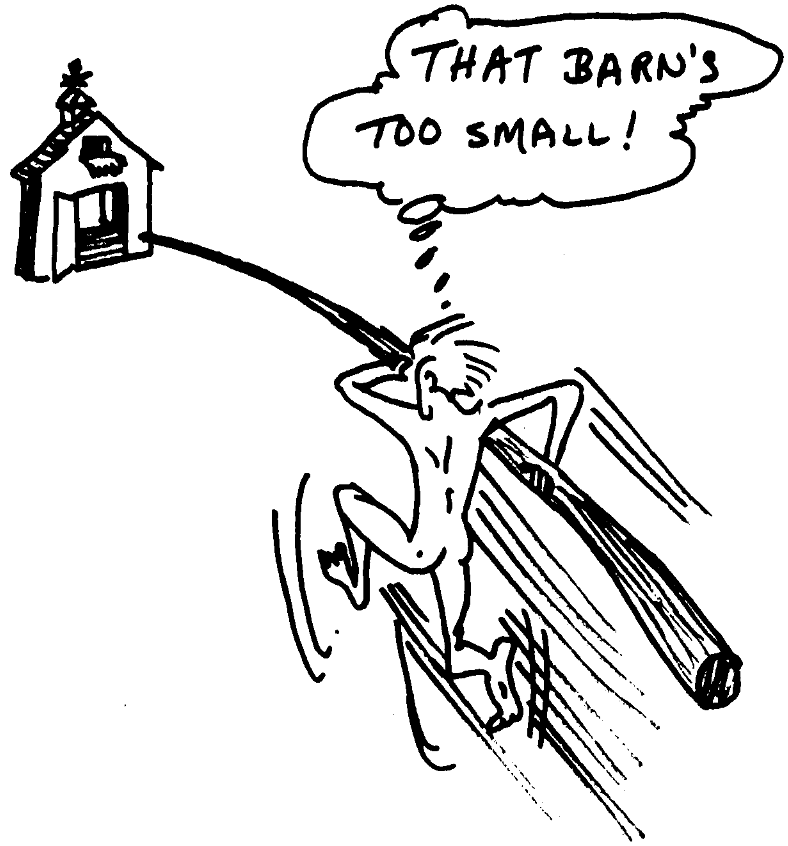
7. Show from the Lorentz transformation equations (15) that two events simultaneous ($t_1=t_2$) in a given frame but separated in space ($x_1 \neq x_2$) are not in general simultaneous in another frame.

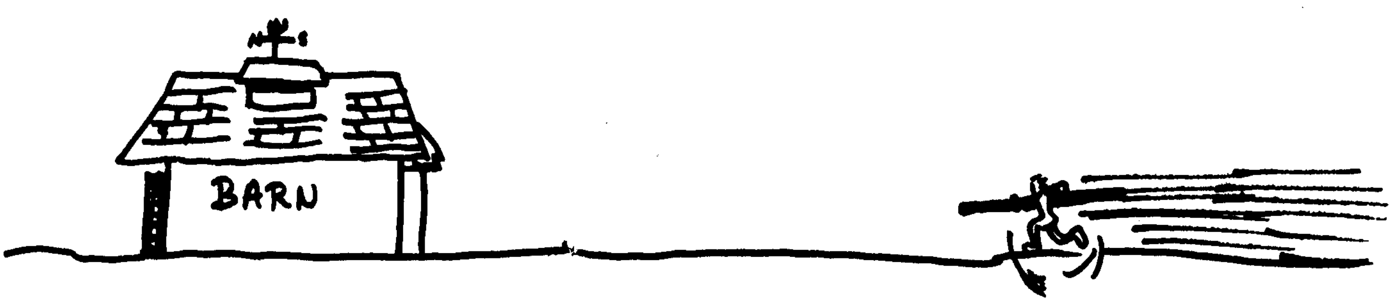
8. Show, using the addition of velocity equations developed in Section VII, that a photon moving with velocity c in frame O' will still be moving with velocity c in another frame moving with respect to frame O'.

9. When Einstein was a boy, he mulled over the following puzzles. A runner looks at himself in a mirror that he holds at arms length in front of him. If he runs with nearly the speed of light, will he be able to see himself in the mirror? Analyze the answer in terms of relativity.

10. (from Spacetime Physics by Taylor and Wheeler.) A worried student writes "Relativity must be wrong. Consider a pole 2L meters long carried so fast in the direction of its length that it appears to be only L meters long in the laboratory frame of reference. Therefore, at some instant the pole can be entirely enclosed in a barn L meters long. However, look at the same situation from the frame of reference of the runner. To him the barn appears contracted to half its length. How can the pole fit into the barn? Does not this unbelievable conclusion prove that relativity contains somewhere a fundamental logical inconsistency?"

Write a reply to the worried student explaining clearly and carefully how the pole and barn are treated by relativity without contradiction.



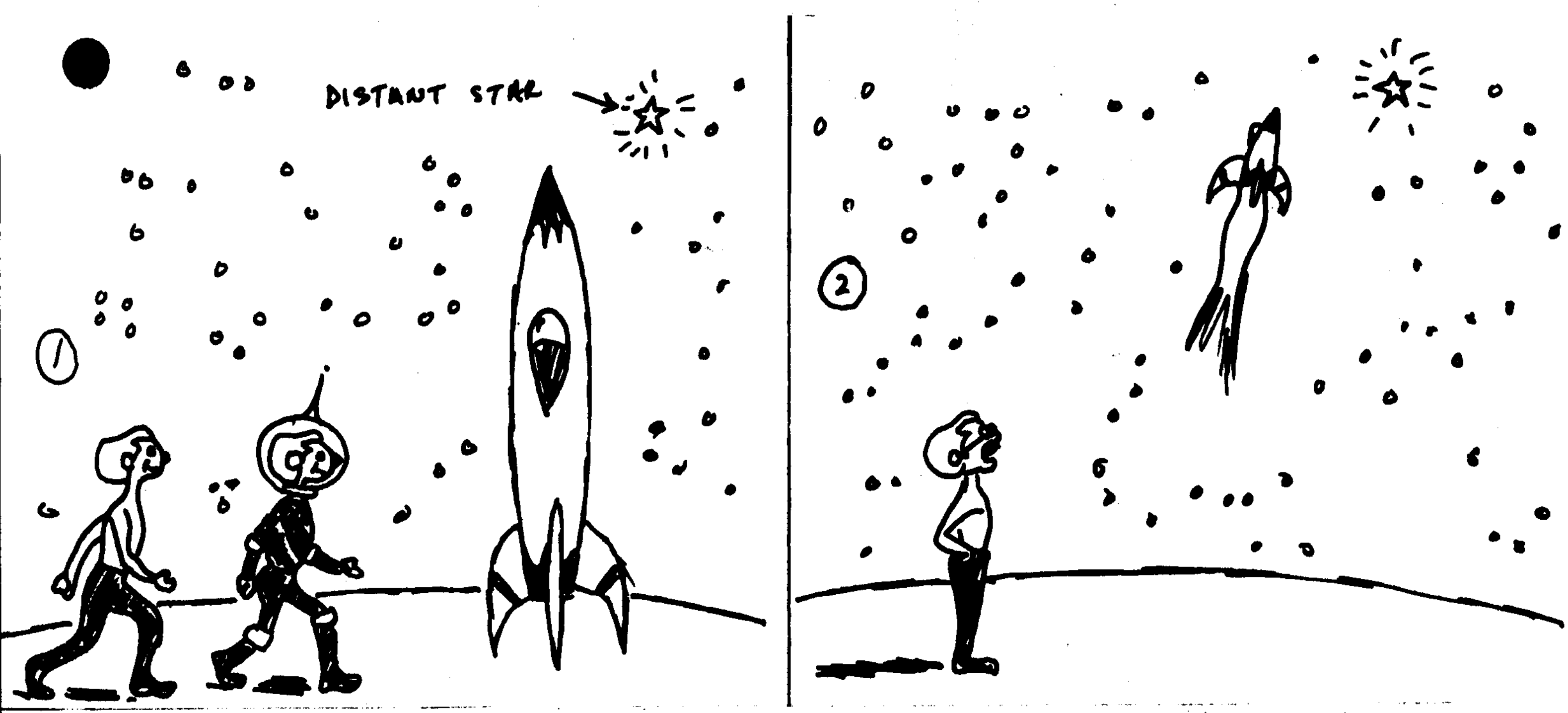


11. The twins paradox. A goes on a trip to ≺ Centarui (L meters away) and back again. He travels at speed v with respect to the earth both ways, and transmits radio signals of frequency f_o in his own rest frame.

The twin "paradox"

For the following remember that when an observer travels towards a source of light (or any electromagnetic radiation) the frequency he observes is shifted and is given by

$$f = \sqrt{\frac{1+\beta}{1-\beta}}\ f_o$$



a) How many signals does A receive before he turns aroung? How many in his return trip? What is then, the total number of signals received by A?

b) How many signals does B receive before A turns around?

c) How many signals does B receive during A's return trip? (Be careful!)

d) What is the total number of signals received by B?

e) Who is younger at the end of the trip?