

P221 Supplementary Note 3

Coupled Oscillators

This is a topic which is not normally covered in a course at this level, but it is a nice illustration of some of the ideas we have been discussing regarding oscillations, interference and beats. So I will boldly go forth and discuss it here. Suppose we have two simple pendula, each consisting of a bob of mass m suspended by a massless rod of length ℓ . The pendula are coupled at their bobs by a massless spring of spring constant k . When the pendula are hanging straight down, the spring connecting them is in its unstretched state. The situation is shown in Fig. 1.

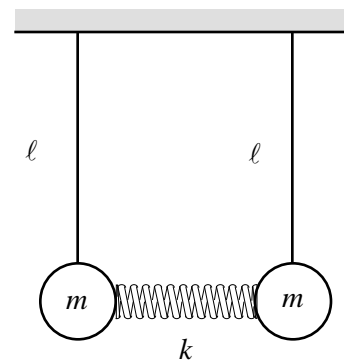


Figure 1

Before we proceed, I would like to call your attention to Fig. 2. Here I show a single simple pendulum. It is displaced by angle θ from its equilibrium position. I can also describe the displacement by x , the displacement from equilibrium. We already know that the differential equation governing displacements in θ can be written as:

$$\frac{d^2\theta}{dt^2} = -\omega_o^2 \theta$$

(1)

where $\omega_o^2 = \frac{g}{\ell}$ and g is the acceleration due to gravity. I can multiply both sides of the above by ℓ to get an expression in x instead since $x = \ell \theta$. This gives us:

$$\frac{d^2x}{dt^2} = -\omega_o^2 x$$

(2)

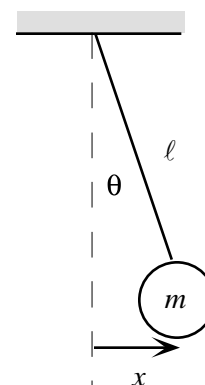


Figure 2

Symmetric Oscillation of the Coupled Pendula

Now refer to Fig. 3. I displace both of our coupled pendula by the same amount and in the same direction. That is: $x_1 = x_2$. The spring is unstretched and remains unstretched as the pendula oscillates. It is as if the spring were not there. Both pendula oscillate with the same frequency, which we call $\omega_S = \omega_o$. The subscript (S) refers to the *symmetric* mode frequency.

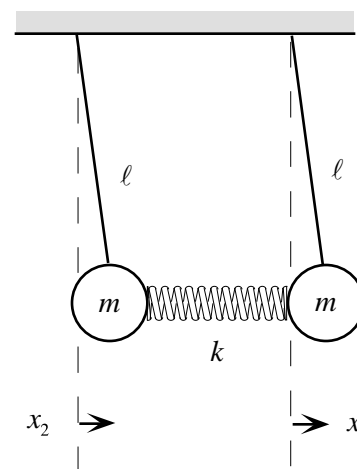


Figure 3

Asymmetric Oscillation of the Coupled Pendula

Now refer to Fig. 4. In this case we have displaced both pendula in equal and opposite directions from their equilibrium positions. That is, $x_1 = -x_2$. Now we cannot ignore the spring since it stretches and compresses as the pendula oscillate. Now let's look at pendulum 1. The restoring force on this pendulum, due to gravity alone is $-m \omega_o^2 x_1$. The force due to the spring is $-2 k x_1$. Why the factor of 2? If we displace pendulum 1 by a certain distance from equilibrium, pendulum 2 is displaced by the same amount in the opposite direction so the spring is stretched by twice the displacement of either pendulum. The force equation for pendulum 1 is:

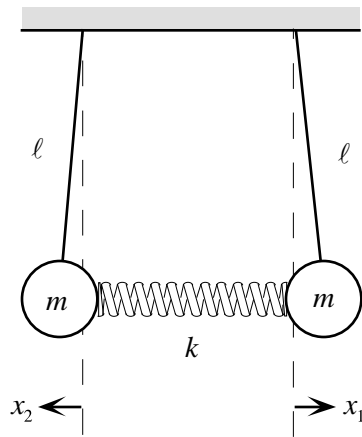


Figure 4

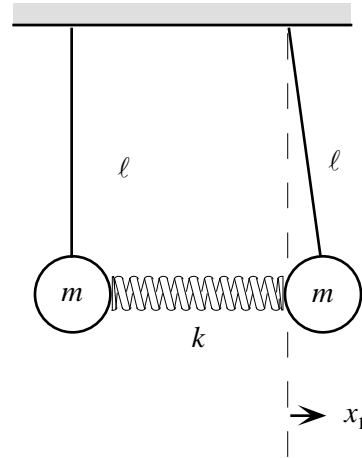


Figure 5

$$m \frac{d^2 x_1}{dt^2} = -m\omega_o^2 x_1 - 2kx_1 \quad (3)$$

or, dividing by mass:

$$\frac{d^2 x_1}{dt^2} = -\omega_o^2 x_1 - \frac{2k}{m} x_1 = -(\omega_o^2 + 2\omega_c^2)x_1 \quad (4)$$

where $\omega_c^2 = k/m$. In this mode, both pendula oscillate with the same frequency, $\omega_A = \sqrt{\omega_o^2 + 2\omega_c^2}$ where the subscript A stands for *asymmetric* and the subscript C for the coupling frequency of the spring.

Other Modes of Oscillation

The symmetric mode and asymmetric modes of oscillation described above are special modes. Clearly they required starting the oscillation in a special way. Each has its own characteristic frequency. Now we will start off the oscillation by holding pendulum 2 fixed and displacing pendulum 1 from its equilibrium position (see Fig. 5). Both pendula are released at rest. What happens now? The subsequent forces acting on the pendula are given by the following equations:

$$m \frac{d^2 x_1}{dt^2} = -m\omega_o^2 x_1 - k(x_1 - x_2) \quad (5)$$

and

$$m \frac{d^2 x_2}{dt^2} = -m\omega_o^2 x_2 - k(x_2 - x_1) \quad (6)$$

Notice that these equations will yield what we had before for the cases: $x_1 = x_2$ and $x_1 = -x_2$. Now add equations (5) and (6) and divide by mass to get:

$$\frac{d^2 (x_1 + x_2)}{dt^2} = -\omega_o^2 (x_1 + x_2) \quad (7)$$

Subtract (6) from (5) and divide by mass to get:

$$\frac{d^2(x_1 - x_2)}{dt^2} = -\omega_o^2(x_1 - x_2) - 2\omega_c^2(x_1 - x_2) \quad (8)$$

Introduce the variables:

$$\begin{aligned} q_1 &= x_1 + x_2 \\ q_2 &= x_1 - x_2 \end{aligned} \quad (9)$$

We can now write equations (7) and (8) in neater form:

$$\frac{d^2 q_1}{dt^2} = -\omega_s^2 q_1 \quad (10)$$

$$\frac{d^2 q_2}{dt^2} = -\omega_o^2 q_2 - 2\omega_c^2 q_2 = -(\omega_o^2 + 2\omega_c^2) q_2 = -\omega_A^2 q_2 \quad (11)$$

Notice this important feature: in the variables x_1, x_2 the force equations are coupled (see equations (5) and (6)) whereas in the variables q_1, q_2 the force equations are decoupled (see equations (10) and (11)).

Let us write the solution to (10) and (11) as:

$$\begin{aligned} q_1 &= A \cos \omega_s t \\ q_2 &= B \cos \omega_A t \end{aligned} \quad (12)$$

where A and B are constants. Equations (9) can be inverted to give:

$$\begin{aligned} x_1 &= \frac{1}{2}(q_1 + q_2) \\ x_2 &= \frac{1}{2}(q_1 - q_2) \end{aligned} \quad (13)$$

Using equations (12), we get:

$$\begin{aligned} x_1 &= \frac{1}{2}(A \cos \omega_s t + B \cos \omega_A t) \\ x_2 &= \frac{1}{2}(A \cos \omega_s t - B \cos \omega_A t) \end{aligned} \quad (14)$$

Now let's recall the special way we want to start these pendula, both at rest initially and initially 1 is displaced by amount C and initially 2 is not displaced. The conditions $\frac{dx_1}{dt}(t=0) = 0$ and $\frac{dx_2}{dt}(t=0) = 0$ are clearly satisfied by equations (14). Using $x_2(t=0) = 0$ in the second of equations (14) gives us $A = B$. Using this in the first of equations (14) with $x_1(t=0) = C$ gives us:

$$\begin{aligned} x_1 &= \frac{C}{2}(\cos \omega_s t + \cos \omega_A t) \\ x_2 &= \frac{C}{2}(\cos \omega_s t - \cos \omega_A t) \end{aligned} \quad (15)$$

Now let me remind you of two trig relations which are useful here:

$$\begin{aligned} \cos\theta_1 + \cos\theta_2 &= 2 \cos \frac{\theta_1 - \theta_2}{2} \cos \frac{\theta_1 + \theta_2}{2} \\ \cos\theta_1 - \cos\theta_2 &= 2 \sin \frac{\theta_1 - \theta_2}{2} \sin \frac{\theta_1 + \theta_2}{2} \end{aligned} \quad (16)$$

Apply these to equations (15):

$$\begin{aligned} x_1(t) &= C \cos \frac{\omega_s - \omega_A}{2} t \cos \frac{\omega_s + \omega_A}{2} t \\ x_2(t) &= C \sin \frac{\omega_s - \omega_A}{2} t \sin \frac{\omega_s + \omega_A}{2} t \end{aligned} \quad (17)$$

Each pendulum oscillates with frequency $\frac{\omega_s + \omega_A}{2}$ and the amplitude is modulated by a sinusoidal varying function of frequency: $\frac{\omega_s - \omega_A}{2}$. Eventually the amplitude of pendulum 1 goes to zero because of the $\cos \frac{\omega_s - \omega_A}{2} t$ factor.

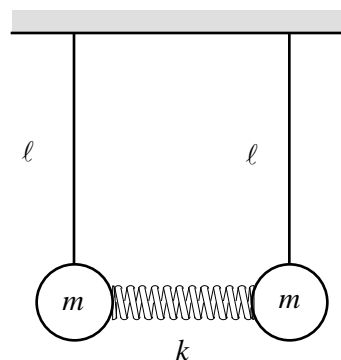
When that happens, the amplitude of pendulum 2 is a maximum and vice-versa. Sometimes pendulum 1 has all the energy and at other times pendulum 2 has all the energy.

This was demonstrated in class and will be studied further using *Interactive Physics*.

Classroom Demonstration



This demonstration was shown in class to illustrate the motion of two coupled pendula. In this apparatus, the pendula are joined by a spring.



Mathematica

Below I show plots of the displacement of our oscillator using *Mathematica*. I chose $f_s = 1.0 \text{ Hz}$ ($\omega_s = 2\pi f_s$) and $f_A = 0.9 \text{ Hz}$ ($\omega_A = 2\pi f_A$). I assume an initial displacement $x_1(t=0) = 1$. The time behavior of the displacements of both pendula is shown. The *Mathematica* commands which generated these plots are:

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Plot[(Cos[2 Pi t] + Cos[2 Pi .9 t])/2,{t,0,20},PlotPoints->50]
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Plot[(Cos[2 Pi t] - Cos[2 Pi .9 t])/2,{t,0,20},PlotPoints->50]
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