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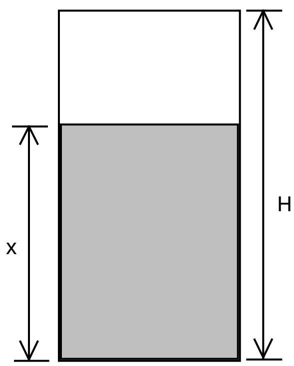
Supplementary Note # 9

Problem 9.12
Leaky Coke Can

Solution to Problem 9.12 in the Text (Resnick & Halliday)

You are asked, in this problem, to consider what happens to the center-of-mass of the liquid and can as the level of liquid in a can leaks out. Refer to the drawing. The mass of the can is M and the mass of the liquid in the can is m. The height of the can is H.

For any given height, x, of liquid in the can, the mass of the liquid is $m\, x/H$ and the center-of-mass, h, is given by:



$$h = \frac{M(H/2) + (m\, x/H)(x/2)}{M + m\, x/H} \qquad (1)$$

Now I will introduce two variables:

$$f = \frac{\text{initial mass of liquid}}{\text{mass of can}} = \frac{m}{M}$$

and

$$z = \frac{\text{height of liquid in can}}{\text{height of can}} = \frac{x}{H}$$

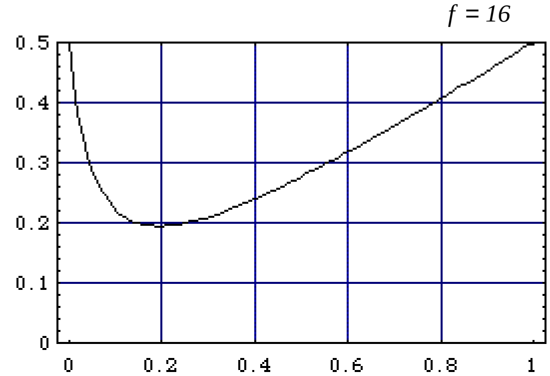
with this, eq (1) becomes:

$$h(z) = \frac{H}{2}\left(\frac{1+fz^2}{1+fz}\right) \qquad (2)$$

Notice that when the can is full ($z = 1$) and when the can is empty ($z = 0$), $h = H/2$ as expected. While the liquid is leaving the can we expect the center of mass to drop below $H/2$. This means that $h(z)$ must have a minimum.

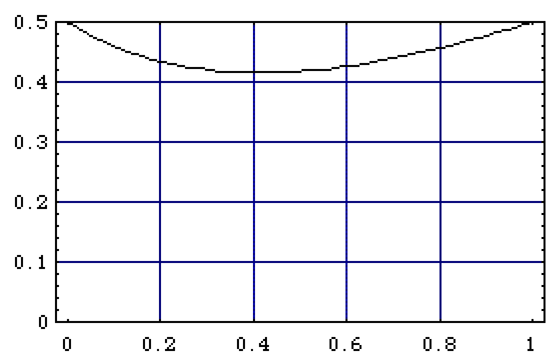
Before we find the minimum, we plot $h(z)$ using *Mathematica*, for a typical 12 0z can of soda pop (for which $f \approx 16$). We also show the plot for $f = 1$. The *Mathematica* commands which produced these plots are given.

f = 16



$$\frac{1}{2}\left(\frac{1+f\,z^2}{1+f\,z}\right)$$

f = 1




```
Plot[ .5 (1+16 z^2)/(1+16
z),{z,0,1},PlotRange->{0.,.5},
PlotPoints->50,Frame->True,GridLines-
>Automatic]
```

```
Plot[ .5 (1+ z^2)/(1+ z),{z,0,1},PlotRange->{0.,.5},
PlotPoints->50,Frame->True,GridLines->Automatic]
```

We will also use *Mathematica* to help us check our algebra as we find out where the minimum occurs and the minimum value of $h(z)$. First, let's find where the minimum occurs:

$$\frac{d}{dz}\left(\frac{1+fz^2}{1+fz}\right)=\frac{2fz}{1+fz}-\frac{(1+fz^2)f}{(1+fz)^2}=0.$$ This leads to the quadratic equation: $f^2z^2+2fz-f=0$

We take the positive (physical) root: $z=\frac{1}{f}\left(\sqrt{1+f}-1\right)$. Substitution into (2) yields:

$$h=\frac{H}{f\sqrt{1+f}}\left[1+f-\sqrt{1+f}\right].$$ Please refer to the *Mathematica* output.

Here we define the function h[z]. We leave off the H/2. Note the syntax.

```
In[3]:=
   h[z_] := (1 + f z^2)/(1 + f z)
In[4]:=
   h[z]
Out[4]=
             2
   1 + f z
   ---------
   1 + f z
```

Verify that the function looks right.

```
In[5]:=
   D[h[z],z]
Out[5]=
                           2
    2 f z     f (1 + f z )
   -------- - -------------
   1 + f z               2
                (1 + f z)
```

The `D[f,x]` *function takes the derivative of* `f` *with respect to* `x`.

```
In[6]:=
   Simplify[%]
Out[6]=
                          2
   f (-1 + 2 z + f z )
   -------------------
                2
       (1 + f z)
```

% refers to the last expression output by Mathematica.

Find the roots of the above.

```
In[7]:=
   Solve[%==0,z]
Out[7]=
            -2   2 Sqrt[1 + f]           -2   2 Sqrt[1 + f]
            -- + -------------           -- - -------------
            f          f                 f          f
   {{z -> ------------------}, {z -> ------------------}}
                   2                          2
In[8]:=
   h[z] /. z->(Sqrt[1+f]-1)/f
Out[8]=
                              2
         (-1 + Sqrt[1 + f])
   1 + ---------------------
                  f
   -------------------------
          Sqrt[1 + f]
```

`/. z->(Sqrt[1+f]-1)/f` *replaces* `z` *by the positive root in the expression* `h[z]`

```
In[9]:=
   Expand[%]
Out[9]=
   -2        2               2
   -- + ----------- + -------------
   f    Sqrt[1 + f]   f Sqrt[1 + f]
```