Lecture 13
Monday - February 20, 2006
Written or last updated: February 19, 2006

P442 – Analytical Mechanics - II
The Tensor of Inertia

Although we covered this in a previous lecture – for completeness I give here the proof for the parallel axis theorem and the perpendicular axis theorem.

Parallel axis theorem

We refer to Figure 1 to prove the parallel-axis theorem in two dimensions. Suppose we have parallel axes one of which passes through point O and the other through point C where the latter is the center of mass of the body. An element of mass m_i within the body is located by X_i, Y_i with respect to the point O or x_i, y_i with respect to the point C where $X_i = x_i + h_x$ and $Y_i = y_i + h_y$. The two axes are separated by h where $h^2 = h_x^2 + h_y^2$.

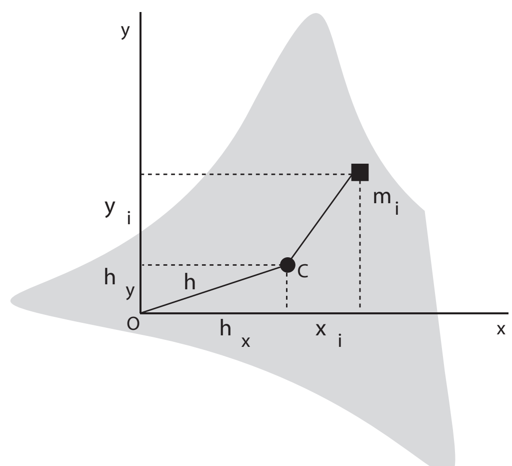

Figure 1: Parallel axis theorem

With respect to the point O the moment of inertia is given by:

$$I_O = \sum_i m_i(x_i + h_x)^2 + \sum_i m_i(y_i + h_y)^2 = \sum_i m_i(x_i^2 + y_i^2) + Mh^2 + 2h_x \sum_i m_i x_i + 2h_y \sum_i m_i y_i \quad (1)$$

In the above, the two rightmost terms separately vanish since the x_i, y_i are with respect to the center of mass. The total mass is $M = \sum m_i$ and the first term of the RHS is just the moment of inertia with respect to the center of mass, I_C. Hence:

$$I_O = I_C + Mh^2 \quad (2)$$

Perpendicular axis theorem

Suppose a lamina lies in the x, y plane. The moment of inertia about the z-axis is just:

$$I_z = \int (x^2 + y^2) dm = \int x^2 dm + \int y^2 dm \tag{3}$$

The first term on the RHS is just the moment of inertia about the y-axis and the second term is just the moment of inertia about the x-axis so:

$$I_z = I_y + I_x \tag{4}$$

Dynamic balancing

Figure 2 shows a square lamina in the x, y plane at $t = 0$. The axis of rotation is the x-axis – not one of the principal axes.

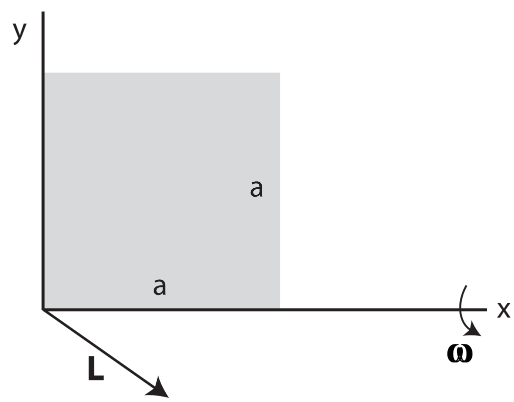

Figure 2: Square lamina.

The angular momentum vector is given by:

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = ma^2 \begin{pmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = ma^2\omega \begin{pmatrix} 1/3 \\ -1/4 \\ 0 \end{pmatrix} \tag{5}$$

As the plate rotates the angular momentum vector $\mathbf{L}$ sweeps out a cone. But that means there is a $d\mathbf{L}/dt$ which means there must be a reaction torque τ perpendicular to the axis of rotation. If an object is *statically* balanced the axis of rotation must pass through the center of mass. To be *dynamically* balanced the axes of rotation must lie along the principal axis.

In our above example what would happen if we moved the origin of coordinates to the center of the square plate (see Figure 3(a)) but still kept the x, y axes parallel to the edges? The new intertia tensor would be:

$$\mathbf{I} = ma^2 \begin{pmatrix} 1/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/6 \end{pmatrix} \tag{6}$$

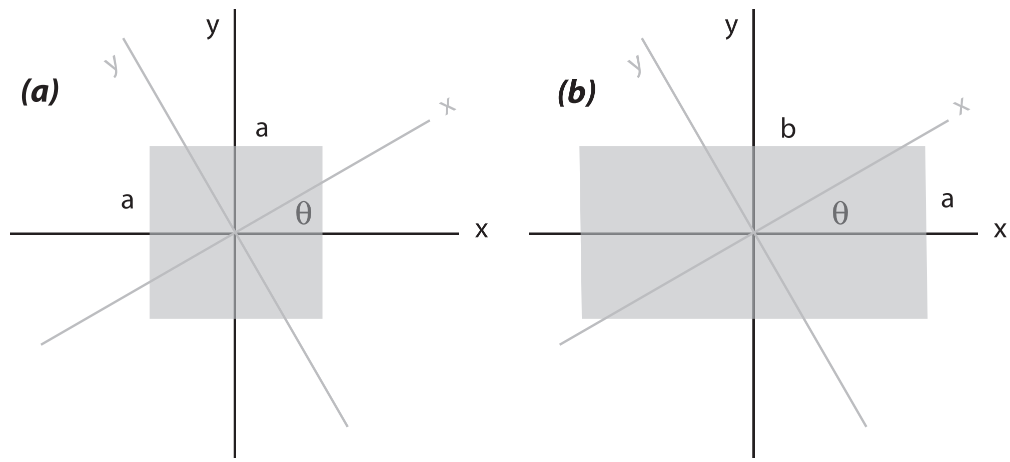

Figure 3: Square and rectangular laminas.

Here is something interesting. For a rectangular lamina with the axis perpendicular to the lamina and through its center of mass and x, y axes parallel to the edges (see Figure 3(b)) we have:

$$\mathbf{I} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A+B \end{pmatrix} \tag{7}$$

where $A = ma^2$ and $B = mb^2$. Obviously the above also works for the square with $A = B$. Now let's rotate the coordinate system about the z-axis through angle θ. What happens to $\mathbf{I}$ in this new system?

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A+B \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$
$$\begin{pmatrix} A\cos^2\theta + B\sin^2\theta & (A-B)\cos\theta\sin\theta & 0 \\ (A-B)\cos\theta\sin\theta & B\cos^2\theta + A\sin^2\theta & 0 \\ 0 & 0 & A+B \end{pmatrix} \tag{8}$$

Notice that if $A = B$ the matrix is diagonal – the products of inertia are are zero – and any perpendicular axes in the $x - y$ are principal axes for the square. But this is not the case for the rectangle. Does this make sense to you?

An aside – note that for the transformed matrix the perpendicular axis theorem still holds and the trace of the matrix was unchanged under the similarity transformation.

An alternative way to find the principal axes.

Suppose we have an object that possesses a symmetry axis (see Figure 4). The other principal axes lie in the $x-y$ plane. Suppose that the x'-axis is a principal axis that it makes angle θ with the unprimed x-axis.

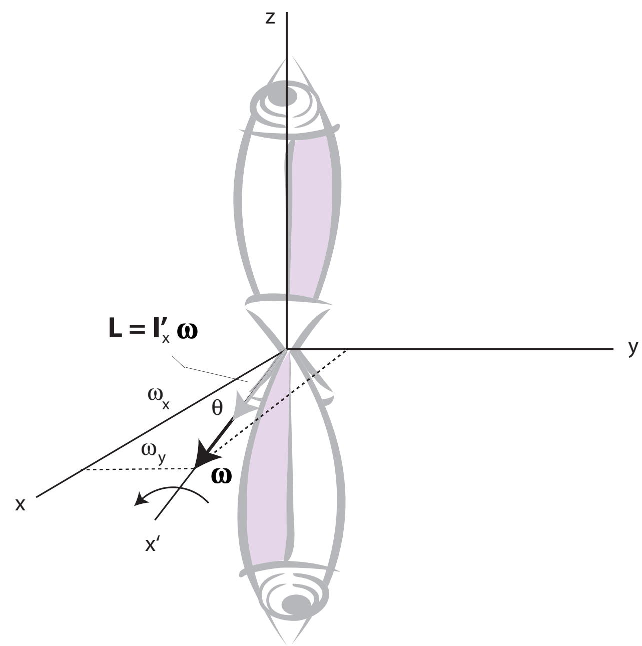

Figure 4: The z-axis is a symmetry axis and therefore a principal axis. The other principal axes lie in the $x-y$ plane. Suppose that x' is a principal axis that we rotate about this axis. Then along this axis, the moment of inertia is I' and the angular momentum vector $\mathbf{L}$ and angular velocity vector $\boldsymbol{\omega}$ are parallel and related by: $L = I'_x \omega$.

Reference frame - We will call the unprimed axes our reference frame. The x, y axes are *not* principal axes but the z-axis is a principal axis. In this reference frame:

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \tag{9}$$

Since the z-axis is a symmetry axis the products of inertia involving z are zero.

Principal axes frame - We now assume that the primed frame is the principal axis frame. In this frame the inertia tensor is represented by a diagonal matrix:

$$\mathbf{I}' = \begin{pmatrix} I'_x & 0 & 0 \\ 0 & I'_y & 0 \\ 0 & 0 & I'_z \end{pmatrix} \tag{10}$$

Note that since in both frames the z axis is a principal axis I_{zz} in equation 9 is equal to I'_z in equation 10.

Rotate around x′ - As shown in Figure 4, we rotate around the x' axis with angular velocity $\boldsymbol{\omega}$. Since this is a principal axis we know that the angular momentum vector $\mathbf{L}$ and the vector $\boldsymbol{\omega}$ must be parallel and furthermore $L = I'_x \omega$.

In the reference frame the components of $\boldsymbol{\omega}$ are ω_x and ω_y and $\tan\theta = \omega_y/\omega_x$. Also, in the reference frame, $L_x = I'_x \omega_x$, $L_y = I'_x \omega_y$ and $L_z = 0$. But also in the reference frame:

$$\mathbf{L} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ 0 \end{pmatrix} = \begin{pmatrix} I_{xx}\omega_x + I_{xy}\omega_y \\ I_{xy}\omega_x + I_{yy}\omega_y \\ 0 \end{pmatrix} = I'_x \begin{pmatrix} \omega_x \\ \omega_y \\ 0 \end{pmatrix} \tag{11}$$

So we have the equations:

$$I'_x \omega_x = I_{xx}\omega_x + I_{xy}\omega_y \tag{12}$$
$$I'_x \omega_y = I_{xy}\omega_x + I_{yy}\omega_y \tag{13}$$

Dividing both by ω_x we get:

$$I'_x = I_{xx} + I_{xy}\tan\theta \tag{14}$$
$$I'_x \tan\theta = I_{xy} + I_{yy}\tan\theta \tag{15}$$

Eliminate I'_x from the above two equations and make use of the trig identity: $\tan 2\theta = 2\tan\theta/(1-\tan^2\theta)$ to finally arrive at:

$$\tan 2\theta = \frac{2I_{xy}}{I_{xx} - I_{yy}} \tag{16}$$

The nice thing here is that you can find the principal axes without having to find the eigenvectors.