P442 – Analytical Mechanics - II
The Tensor of Inertia

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Diagonalization of matrices

The matrix representing the inertia tensor can be put in diagonal form and we will examine the physical consequences below but here I remind you about the procedure for diagonalizing a matrix. You may have your favorite reference for how to go about doing this, as I do\textsuperscript{1}.

We assume that in some coordinate system we have a \( N \times N \) matrix \( A \) and that there exists vectors \( x \) such that the vector \( Ax \) is just a multiple of \( x \). This is usually written in the form:

\[
Ax = \lambda x \tag{1}
\]

A vector that satisfies the above is called an \textit{eigenvector} and the corresponding number \( \lambda \) is called an \textit{eigenvalue}. In general there are \( N \) eigenvectors and \( N \) eigenvalues but these eigenvalues are not necessarily distinct. For the matrices we will be dealing with the eigenvalues are real and the eigenvectors are orthogonal. Our inertia matrices satisfy this since they be hermetian, that is, equal to the complex conjugate of its transpose.

How to find the eigenvalues and eigenvectors? For equation 1 to be satisfied for non-zero vectors we require that:

\[
|A - \lambda 1| = 0 \tag{2}
\]

where \( 1 \) is the unit matrix. The above equation is called the \textit{characteristic equation} and the LHS of the above equation is called the \textit{characteristic or secular} determinant. The roots of the above equation are the eigenvalues.

Once the eigenvalues are determined, the components of the eigenvectors are found for each eigenvalue in the following way: for a particular eigenvalue the equation \( Ax = \lambda x \) implies \( N \) linear equations in for the \( N \) coordinates. Once those coordinates are determined we also make one other requirement, that the eigenvector be normalized so that the sum of the squares of its components add to give unity.

Using \textit{Mathematica} You can use \textit{Mathematica} to do the grunge work for you. For example, consider the matrix:

\[ a = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \]

*Mathematica* will return the eigenvalues for you by using the command: \texttt{Eigenvalues[a]} and in this case the result comes back as \{-6, 2, 3\}. Or you could ask *Mathematica* to expand the determinant \(|A - \lambda I|\) for you and get back \(\lambda^3 + \lambda^2 - 24\lambda + 36\) and setting this equal to zero and solving for the roots you get back the eigenvalues.

Consider the three components of the eigenvector associated with the eigenvalue \(\lambda = -6\). These would satisfy:

\[
\begin{align*}
  x_1 + x_2 + 3x_3 &= -6x_1 \\
  x_1 + x_2 - 3x_3 &= -6x_2 \\
  3x_1 - 3x_2 - 3x_3 &= -6x_3
\end{align*}
\]

and similarly for the eigenvalues. Or you could simply issue the *Mathematica* command \texttt{Eigenvectors[a]} and get back \((-1, 1, 2), (1, 1, 0), (-1, -1, 1)\). *Mathematica* does not normalize the eigenvectors. To turn these into normalized eigenvectors you would multiply these three vectors by \(1/\sqrt{6}, 1/\sqrt{2}\) and \(1/\sqrt{3}\) respectively.

**Now the diagonalization**  For a given matrix \(A\), if we construct the matrix \(S\) that has the normalized eigenvectors of \(A\) as its columns, then the similarity transformation \(S^{-1}AS\) will render \(A\) diagonal with the eigenvalues of \(A\) being the elements of the new matrix.

For the example we have been dealing with, here is the matrix \(S\):

\[
S = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix}
\]

and

\[
\begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\]

A similarity transformation leaves the transpose of a matrix (sum of the diagonal elements) unchanged and since the diagonalized matrix has the eigenvalues as the diagonal elements, the sum of the eigenvalues must then be equal to the trace of the original matrix (\texttt{TrA}). This is a good diagnostic check to make sure you found the eigenvalues correctly.
Back to Mathematica  For completeness, *Mathematica* can find the determinant of a matrix (\texttt{Det[a]}), the transpose of a matrix (\texttt{Transpose[a]}) and the trace of a matrix (\texttt{Tr[a]}). And also be aware of the syntax for multiplying matrices. Suppose you have defined the matrices \(a, b\) and \(c\) then their product is written as \(a \cdot b \cdot c\) where a period is used to indicate multiplication.

Existence of principal axes for a rigid body

**Important**

For any rigid body and any point \(O\) in the body there are three orthogonal *principal* axes in the body for which the the matrix representing the inertia tensor is diagonal. When the angular velocity vector \(\omega\) points along any of these axes, so does the angular momentum vector \(L\). The elements of the diagonalized matrix for the inertia tensor are the *principal* moments.

Back to the lamina

In the previous lecture we considered a square lamina of side \(a\) lying in the \(x - y\) plane with its corner at the origin. The inertia matrix is:

\[
I = \begin{pmatrix}
\frac{1}{3} & -\frac{1}{4} & 0 \\
-\frac{1}{4} & \frac{1}{3} & 0 \\
0 & 0 & \frac{2}{3}
\end{pmatrix}
\]

The eigenvalues for this matrix are 1/12, 7/12 and 2/3. The normalized eigenvectors are:

\[
e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

The rotation matrix that will take us from the original coordinate axes to these axes is:

\[
S = \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

The principal axes are along the diagonal of the square lamina and the another axis in the \(x - y\) plane perpendicular to that diagonal and the original \(z\) axis. For these principal axes as the coordinate system the diagonalized inertia matrix is:

\[
I = \begin{pmatrix}
\frac{1}{12} & 0 & 0 \\
0 & \frac{7}{12} & 0 \\
0 & 0 & \frac{2}{3}
\end{pmatrix}
\]
It the lamina rotates about the \( z \) axis with angular velocity \( \omega \), then the angular momentum points along the \( z \) axis and has magnitude \( 2\omega/3 \). If it rotates along the diagonal of the lamina then \( L = \omega/12 \).