Lecture 4
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P442 – Analytical Mechanics - II
Relativistic Collisions

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Four-vector momentum and conventions

We will discuss the subject of collisions within the context of special relativity. And we start off by reminding you about some important results from special relativity. Figure 1 shows a starred inertial frame moving along the z axis of an unstarred inertial frame with velocity v with respect to the unstarred frame. Four-vectors in the two frames are related by a Lorentz transformation that maintains the magnitude of the four-vector invariant in the same same way that rotations leave the length of three-vectors invariant. And to continue the analogy the dot product (scalar product) of two four-vectors is a Lorentz invariant just as the scalar product of two three-vectors is left unchanged under a rotation of coordinates.

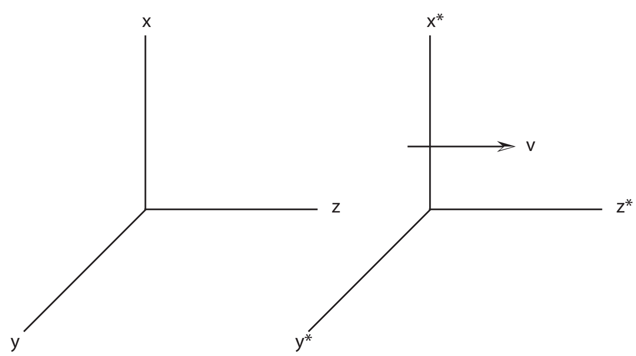

Figure 1: Two inertial frames, one moving with constant velocity v along the z-axis with respect to the other.

The four-vector of particular interest for us is the energy-momentum four-vector of a particle of mass m which we write as $p = (E, p_x, p_y, p_z) = (E, \mathbf{p})$. I will use p for the momentum four-vector and $\mathbf{p}$ for the so-called three-momentum (p_x, p_y, p_z). The dot product of a the momentum four-vector with itself is a Lorentz scalar and can be written as $p \cdot p = E^2 - \mathbf{p}^2 = m^2$. This quantity is the same in any Lorentz frame and m is rest mass of the particle as measured in a frame in which the particle is at rest. The rest mass of the photon is zero so the magnitude of its four vector is always zero and $E = |\mathbf{p}|$ for the photon. The product of two four-momenta, which is also a Lorentz invariant, can be written as $p_1 \cdot p_2 = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2$

You may be used to seeing this relationship in textbooks: $E = \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$ – but we choose units for energy, momentum and mass that allow us to leave c out of these expressions. These units are, respectively: eV, eV/c and eV/c^2. You will recall that 1 eV (electron volt) equals 1.6×10^{-19} J. Units that are more appropriate for discussing masses of elementary particles and momenta and energies of current and planned accelerators are MeV, GeV or TeV which stand for *mega*-electron volts (10^6 eV), *giga*-electron volts (10^9 eV) and *tera*-electron volts (10^{12} eV) respectively.

Particle masses: In this note we will be referring to the mass of the proton ($m_p = 934$ MeV/c^2), neutral pion ($m_{\pi^0} = 135$ MeV/c^2), muon ($m_\mu = 105$ MeV/c^2) and electron ($m_e = 0.5$ MeV/c^2). We will also sometimes approximate the mass of the proton as ≈ 1 GeV/c^2.

When is a particle relativistic? As a rough guide, when the momentum of the particle is several percent of a rest mass of the particle, relativistic effects start becoming important. Since $\gamma = E/m = \sqrt{p^2 + m^2}/m$ if p can be ignored compared to m then $\gamma = 1$ and when p is a few percent of m we can make the approximation (using the binomial expansion) $\gamma \approx 1 + p^2/2m^2$. This gives you an idea for how quickly relativistic effects turn on as the momentum increases. For example, an electron with momentum 10 MeV/c^2 is relativistic since it has $\gamma \approx 20$ while a proton of the same momentum can probably be treated non-relativistically since it has a $\gamma \approx 1.005$.

Kinetic energy Energy (E), mass (m) and kinetic energy (T) have the following relationship: $E = T + m$. It will be left as an exercise for you to show that with this definition, the non-relativistic approximation for T is $p^2/2m$.

Lorentz transformations

The energy-momentum four-vector transforms like so:

$$\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E^* \\ p_x^* \\ p_y^* \\ p_z^* \end{pmatrix} \tag{1}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1 + \beta^2}$.

Carrying out the matrix multiplication:

$$E = \gamma(E^* + \beta p_z^*); \quad p_x = p_x^*; \quad p_y = p_y^*; \quad p_z = \gamma(p_z^* + \beta E^*) \tag{2}$$

with the familiar result that components of the four-vector transverse to the Lorentz boost direction are unchanged.

Suppose that a particle of rest mass m is at rest in the starred frame. Then its momentum is zero and $E^* = m$. In the unstarred frame the particle has energy $E = \gamma m$ and momentum $p = \beta\gamma m = \beta E$ along the z direction. So we have this important result:

$$\gamma = \frac{E}{m} \quad \text{and} \quad \vec{\beta} = \frac{\mathbf{p}}{E} \tag{3}$$

So if you know the momentum and energy of a particle in a given frame, the above provides you with the β and γ of the particle.

Particle lifetime Many elementary particles are unstable with lifetimes that are determined by the underlying force that determines the decay process – the stronger the force, the shorter the lifetime. The electron and proton are stable particles. The experimental limits on their lifetimes are 10^{26} yr and 10^{32} yr respectively. The lifetime of the muon (a heavier version of the electron) is 2.2×10^{-6} s. The lifetime of the neutron, which is slightly more massive than the proton, is 885 s. The muon and electron (and other leptons) carry a quantum number called *lepton quantum number* that has to be conserved in any process. Since there is no particle lighter than the electron that carries this quantum number, it cannot decay. The muon can decay into an electron plus other particles (actually two neutrinos) because it has enough mass and therefore it is unstable. The proton and neutron (and other baryons) carry a quantum number called *baryon number* which similarly has to be conserved. The proton is the lightest baryon, hence it is stable.

The lifetimes above are given as a mean lifetime τ. The probability that a particle will survive a time t after being produced is $e^{-t/\tau}$.

The lifetime only makes sense in the frame in which the particle is at rest. In a frame in which the particle is moving, the particle's lifetime is dilated by the factor γ and the distance the particle travels in the frame in which it is moving is $\gamma\beta c\tau = pc\tau/m$. For example, at Femilab, muons can be produced with momenta up 800 GeV/c. For these muons $\gamma \approx 8000$. We can safely say that $\beta \approx 1$ and without the time dilation factor the muons would travel a mean distance, before decaying, of $c\tau = 660$ m. With the dilation factor the distance is 5,280 km. Of course, to the muon that distance is shortened by the factor $1/\gamma$ – in its frame it travels 660 m.

Two-body scattering

Consider the scattering process $1 + 2 \to 3 + 4$ in the overall *CMS* as shown in the right-hand portion of Figure 2.

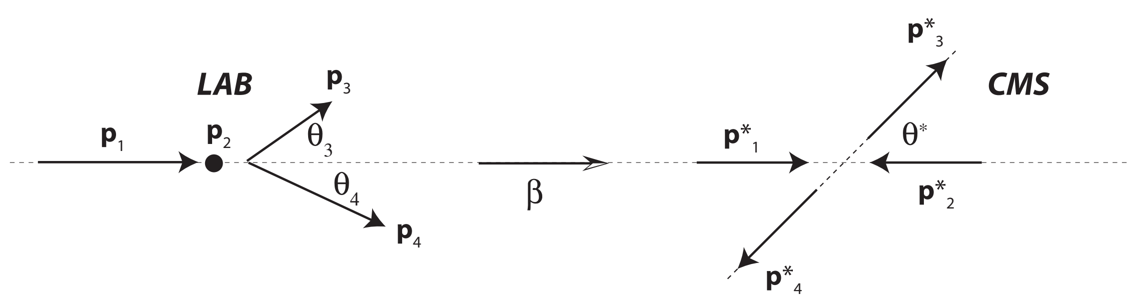

Figure 2: Two-body scattering viewed in the *LAB* and *CMS* frames.

We are going to allow for the possibility that masses change so that masses after the scattering are not necessarily the same as before the scattering. In the *CMS* the sum of the four vectors is given by:

$$p_0^* = p_1^* + p_2^* = (E_1^* + E_2^*) - (\mathbf{p}_1^* + \mathbf{p}_2^*) = E_1^* + E_2^* \tag{4}$$

The three-vector momenta sum to zero in the *CMS*. In the above equation we introduce the total four-momentum in the *CMS* – p_0^* which must also be equal to $p_3^* + p_4^*$. Since the square of the momentum four-vector is the square of the mass of a particle with that four-momentum, we will call the magnitude of p_0^* the mass m_0. You can picture the collision process in the *CMS* as particles 1 and 2 coming together to momentarily form a particle of mass m_0 in the *CMS* that then decays into particles 3 and 4. This will be a useful picture when we look at a particle in its rest frame decaying into two other particles and then viewing this decay in a frame where the particle is moving.

Obviously we have:

$$E_1^* + E_2^* = m_0 = E_3^* + E_4^* \tag{5}$$

In one of the problems below you are asked to show the following:

$$E_1^* = \frac{m_0^2 + m_1^2 - m_2^2}{2m_0} \quad \text{and} \quad E_2^* = \frac{m_0^2 + m_2^2 - m_1^2}{2m_0} \tag{6}$$

$$E_3^* = \frac{m_0^2 + m_3^2 - m_4^2}{2m_0} \quad \text{and} \quad E_4^* = \frac{m_0^2 + m_4^2 - m_3^2}{2m_0} \tag{7}$$

And it is also obvious that $|\mathbf{p}_i^*| = \sqrt{E_i^{*2} - m_i^2}$ where i goes from 1 to 4 and $\beta_i^* = |\mathbf{p}_i^*|/E_i^*$.

In the *LAB* frame the initial total three-momentum is $\mathbf{p}_1$ since $\mathbf{p}_2 = 0$ and the initial total energy is $E_1 + m_2$. In the *LAB* frame the *CMS* is moving with velocity β and it, and its associated γ are given by:

$$\beta = \frac{\mathbf{p}_1}{E_1 + m_2} \quad \text{and} \quad \gamma = \frac{E_1 + m_2}{m_0} \tag{8}$$

It is useful to remind yourself, that viewed from the *CMS* frame the *LAB* frame is moving away with velocity $-\beta$. You should also remind yourself that:

$$(p_1 + p_2)^2 = (p_1^* + p_2^*)^2 = m_0^2 \tag{9}$$

since the squares of four-vectors are preserved in going from one inertial frame to another and evaluating in the *LAB* frame:

$$m_0^2 = m_1^2 + m_2^2 + 2m_2E_1 \tag{10}$$

Fixed target versus collider experiments. In high energy experiments, especially those whose goal is to discover massive new particles, like the Higgs particle, total available energy is important. One way to do the experiment is to have a beam of particles with a certain energy hit target particles (like protons in liquid hydrogen) at rest. The other technique is to bring two beams of particles of equal energy into head-on collisions. This is the difference between fixed-target and collider experiments. From an energy point of view the collider approach provides more energy for discovery. For example, the new Large Hadron Collider (LHC) at CERN will bring 7 TeV protons in collision with 7 TeV protons. The total available energy is 14 TeV. If a single 7 TeV beam collides with protons at rest the energy available is $\sqrt{2m_p(m_p + E)}$. Using this formula and setting $E = 7$ TeV and being careful to express the proton mass in units of TeV ($m_p = 0.001$ TeV) we get a total available energy of 0.12 TeV, less than 1% of what we get in the collider.

Comparing the *CMS* and *LAB* quantities

Referring to Figure 2 and using our transformation equations 2, for particle 3:

$$\frac{p_{3x}}{p_{3z}} = \tan\theta_3 = \frac{p_3^* \sin\theta^*}{\gamma(p_3^* \cos\theta^* + \beta E_3^*)} \tag{11}$$

where β is the velocity of the *CMS* as given by equation 8. Dividing numerator and denominator of the last term by p_3^* we get:

$$\tan\theta_3 = \frac{\sin\theta^*}{\gamma(\cos\theta^* + \beta/\beta_3^*)} \tag{12}$$

For the non-relativistic case ($\gamma = 1$) this is what we obtained in discussing collisions without relativity. Again, invoking our Lorentz transformation:

$$\frac{p_3 \cos\theta_3}{E_3} = \beta_3 \cos\theta_3 = \frac{\gamma(p_3^* \cos\theta^* + \beta E_3^*)}{\gamma(E_3^* + \beta p_3^* \cos\theta^*)} \tag{13}$$

After some simplifying:

$$\beta_3 \cos\theta_3 = \frac{\beta_3^* \cos\theta^* + \beta}{1 + \beta\beta_3^* \cos\theta^*} \tag{14}$$

We assume that any angle in the *CMS* is possible, that is, $0 < \theta^* < \pi$. If all the m_i are equal, no particle can go backwards in the *LAB*. If $m_1 = m_3$ and $m_2 = m_4$ and $m_2 > m_1$, then in the *LAB* frame the heavier particle has a maximum angle of $\pi/2$. See one of the problems below.

Decay of the neutral pion

The neutral pion (π^0) has a mass of 0.135 GeV/c^2 and a mean lifetime of about 10^{-16} s. Including time dilation, even a 1 TeV π^0 would travel less than 200 microns before decaying. We can assume, to within a good approximation, that they travel hardly any distance before they decay. And 99% of the time they decay into two photons. Figure 3 shows the decay viewed in the rest frame of the π^0 and also viewed in the frame where the π^0 momentum $\mathbf{p}_\pi$. We refer to this latter frame as the *LAB* frame.

Since photons are massless their energies and magnitudes of their three-momenta are equal. In the rest frame of the π^0 we have $\beta_1^* = \beta_2^* = 1$ and in the *LAB* frame $\beta_1 = \beta_2 = 1$. Also in the rest frame, $E_1^* = E_2^* = m_\pi/2$.

The π^0 carries no spin – so in its rest frame there is no special direction. If we consider any one of the photons, it will come off in any direction with equal probability. If we were to plot the number distribution of photons in some interval of $\cos\theta^*$ ($dN/d\cos\theta^*$) from $\cos\theta^* = +1$ to $\cos\theta^* = -1$ it would be flat. Since the integral of $dN/d\cos\theta^*$ over the full range of $\cos\theta^*$ is usually normalized to 1 then $dN/d\cos\theta^* = 1/2$. Obviously we only have to consider one of the photons because we know the other has to be collinear with it.

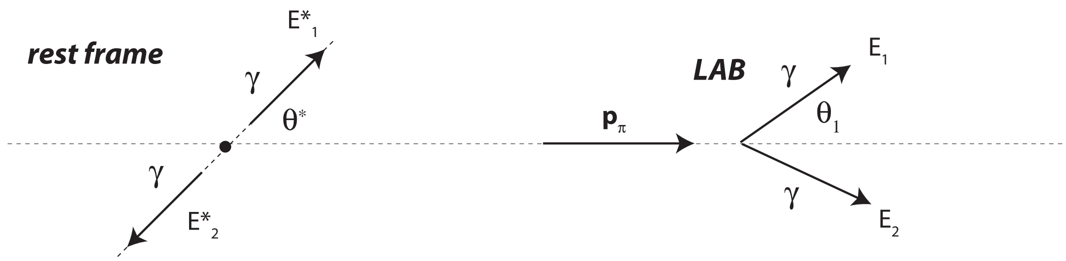

Figure 3: The decay $\pi^0 \rightarrow 2\gamma$ viewed in the rest frame of the π^0 and also viewed in the frame where the π^0 momentum $\mathbf{p}_\pi$.

In the LAB frame we know how to find β_π and γ_π. In that frame, the energy of one of the photons is:

$$E_1 = \gamma_\pi(E_1^* + \beta_\pi E_1^* \cos\theta^*) = \gamma_\pi \frac{m_\pi}{2}(1 + \beta_\pi \cos\theta^*) \tag{15}$$

We can see that the minimum and maximum values of the photons in the LAB are:

$$E_1^{min} = \gamma_\pi \frac{m_\pi}{2}(1 - \beta_\pi) \quad \text{and} \quad E_1^{max} = \gamma_\pi \frac{m_\pi}{2}(1 + \beta_\pi) \tag{16}$$

and we also see that:

$$E_1^{max} - E_1^{min} = \gamma_\pi m_\pi \beta_\pi = p_\pi \tag{17}$$

What about the distribution of LAB energy E_1? From equation 15 we have:

$$dE_1 = \gamma_\pi \frac{m_\pi}{2} \beta_\pi d\cos\theta^* \tag{18}$$

Using the result quoted earlier – $dN/d\cos\theta^* = 1/2$, we conclude that

$$\frac{dN}{dE_1} = \frac{1}{\beta_\pi \gamma_\pi m_\pi} = \frac{1}{p_\pi} \tag{19}$$

But the range of energies allowed is p_π so the integral of dN/dE_1 over that range is one – and that's a good thing.

Distribution in opening angle of the photons in the *LAB*. To find the angle of one of the photons in the LAB, measured with respect to the line-of-flight of the pion, we can borrow equation 14 realizing that the $\beta = 1$ for photons (in any frame).

$$\cos\theta_1 = \frac{\cos\theta^* + \beta_\pi}{1 + \beta_\pi \cos\theta^*} \tag{20}$$

From the above, if one of the photons in the *CMS* has $\cos\theta^* = +1$ the other has $\cos\theta^* = -1$ and these two photons come off with angles 0 and π in the *LAB* respectively – the opening angle is π.

You can convince yourself that the minimum opening angle occurs when the photons come off at $\theta^* = \pi/2$ in the *CMS* and for that situation the *LAB* energies of the photons are equal (see equation 15).

Here is another way to look at the distribution in opening angle between the photons in the *LAB*. Let's call that angle ψ. Starting with the conservation four-momentum in the decay $p_\pi = p_1 + p_2$ we take the squares of these four-vectors:

$$m_\pi^2 = p_\pi^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = 2E_1E_2(1 - \cos\psi) \tag{21}$$

As a reminder, for the photons $p_1^2 = p_2^2 = m_\gamma^2 = 0$. After a little re-arranging and a little trig:

$$\sin\frac{\psi}{2} = \frac{m_\pi}{2\sqrt{E_1E_2}} \tag{22}$$

For the minimum opening angle case $E_1 = E_2 = E_\pi/2$ we find:

$$\sin\frac{\psi_{min}}{2} = \frac{1}{\gamma_\pi} \tag{23}$$

In one of the problems you are asked to find the minimum opening angle by differentiating equation 22. And you will also be asked to find the formula for the distribution in the opening angle ψ.