P441 – Analytical Mechanics - I
Circular Orbits and their Stability

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Escape velocity

Let us start with escape velocity from the Earth. By *escape velocity* we mean the velocity required to escape the influence of the Earth's gravitational field. The total energy of a rocket of mass mass m and velocity v a distance r away from the center of the Earth (mass M) is:

$$E_t = T + U = \frac{1}{2}mv^2 - \frac{GMm}{r} \tag{1}$$

Suppose we launch a rocket from the surface of the Earth (radius R_E) with sufficient velocity v_e so that when it is infinitely far from the Earth, where the potential energy is zero, the kinetic energy is also zero and therefore the total energy is also zero. If you were to launch with velocity greater than v_e it would still have some velocity at infinity and therefore the total energy E_t would be greater than zero. So the minimum escape velocity at Earth's surface means that $E_t = 0$. That leads us to:

$$\frac{mv_e^2}{2} = \frac{GMm}{R_E} \Rightarrow v_e = \sqrt{2}\sqrt{\frac{GM}{R_E}} \tag{2}$$

The escape velocity is about 11,200 m/s or 25,000 mph.

Circular orbits

Now let us consider a satellite in a circular orbit around the Earth. The centripetal acceleration is v^2/r and since $F = ma$ where the force is the gravitational force:

$$\frac{mv^2}{r} = \frac{GMm}{r^2} \Rightarrow mv^2 = \frac{GMm}{r} \Rightarrow v = \sqrt{\frac{GM}{r}} \tag{3}$$

So this tells us that for a circular orbit the kinetic is half of the negative of the potential energy or $T = -U/2$. It turns out that for elliptical orbits this still holds but for the these quantities are replaced by their averages.

For a satellite or the space shuttle in a so-called *low Earth orbit* we have $r \approx R_E$ and the velocity is the escape velocity divided by $\sqrt{2}$. The period works out to be about 84 min. A satellite in a geosynchronous orbit circles the Earth once a day so that the satellite stays above the same point on the Earth's surface. The latter requirement also means the orbit has to be in the equatorial plane. The height above the Earth surface for such an orbit is about 36,000 km (compared to about 300 km ($r = 1.05R_E$) for the space shuttle).

Stability of circular orbits

Here we answer the following question: suppose a particle is in a circular orbit about the origin of coordinates where r measures distance from the center of the circle and the force is given by:

$$F = -\frac{K}{r^n} \tag{4}$$

where K is a constant. What is the requirement on n such that stable circular orbits exist? What does that mean? If the particle is given a slight nudge toward or away from its circular orbit will the net force be a restoring force.

Since the force, as written above, is a conservative force we can define a potential energy $U(r)$ given by:

$$U(r) = -\int_{\infty}^{r} F(r)dr = -\frac{K}{(n-1)r^{n-1}} \tag{5}$$

The motion of the particle under a central force conserves angular momentum L and, just as we did in an earlier lecture, we can write an effective potential energy:

$$U_{eff}(r) = -\frac{K}{(n-1)r^{n-1}} + \frac{L^2}{2mr^2} \tag{6}$$

From this effective potential energy we can find the effective force:

$$F_{eff} = -\frac{\partial U_{eff}(r)}{\partial r} = -\frac{K}{r^n} + \frac{L^2}{mr^3} \tag{7}$$

The second term is just the centrifugal force (note that $L = mvr$ so the second terms is just mv^2/r). When we have a circular orbit for $r = \rho$ we have $F_{eff} = 0$. And you can convince yourself that for $r < \rho$ we get $F_{eff} > 0$ and for $r > \rho$ we get $F_{eff} < 0$ so deviations from a circular orbit pull it back to a circular orbit.

From equation 7 we find that for a circular orbit:

$$\rho^{n-3} = \frac{mK}{L^2} \tag{8}$$

The condition for a stable orbit is that:

$$\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r=\rho} > 0 \tag{9}$$

But that implies:

$$-\frac{nK}{\rho^{n+1}} + \frac{3L^2}{m\rho^4} > 0 \tag{10}$$

or:

$$(-n+3)\frac{L^2}{m} > 0 \tag{11}$$

or $n < 3$.

More on stability of circular orbits

This is a continuation of Lecture 21 but now we will not restrict ourselves to forces of the form $F(r) = -K/r^n$. Now we will allow for any central force and see what the conditions are for a stable circular orbit.

Again, for a particle moving under a central force, two dimensions suffice to locate the particle, r and θ. The velocity vector is given by:

$$\vec{v} = \frac{dr}{dt}\hat{\mathbf{e}}_r + r\frac{d\theta}{dt}\hat{\mathbf{e}}_\theta \tag{12}$$

and the acceleration vector by:

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{e}}_r + \frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\hat{\mathbf{e}}_\theta \tag{13}$$

The equation of motion is $\vec{F} = m\vec{a}$. For a central force $F_\theta = 0$ which means that:

$$\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = 0 \tag{14}$$

which means that the angular momentum $L = mr^2\dot{\theta}$ is a constant.

So for our central force $F(r)$ we have:

$$F(r) = m(\ddot{r} - r\dot{\theta}^2) = m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right) \tag{15}$$

We make the substitution: $g(r) = -F(r)/m$ so equation 15 becomes:

$$-g(r) = \ddot{r} - \frac{L^2}{m^2r^3} \tag{16}$$

We make the assumption that we have a circular orbit of radius ρ and consider an orbit that oscillates about this circle, so that $r = \rho + x$ where x is a small perturbation about the circle. But when $r = \rho$ ($x = 0$):

$$-g(\rho) = -\frac{L^2}{m^2\rho^3} \tag{17}$$

When $x \neq 0$ equation 16 becomes:

$$-g(\rho + x) = \ddot{x} - \frac{L^2}{m^2(\rho + x)^3} = \ddot{x} - \frac{L^2}{m^2\rho^3}\left(1 + \frac{x}{\rho}\right)^{-3} \tag{18}$$

Now make use of the fact that $x/\rho << 1$ and do a Taylor series expansion of the LHS and a binomial expansion of the RHS:

$$g(r) \approx g(\rho) + g'(\rho) \cdot x \tag{19}$$

$$\left(1 + \frac{x}{\rho}\right)^{-3} \approx 1 - \frac{3x}{\rho} \tag{20}$$

Using equations 17, 19 and 20 in 18 gets us this:

$$\ddot{x} + \left(\frac{3g(\rho)}{\rho} + g'(\rho)\right) x = 0 \tag{21}$$

This looks like the differential equation for a simple harmonic oscillator $\ddot{x} + \omega^2 x = 0$ where:

$$\omega^2 = \frac{3g(\rho)}{\rho} + g'(\rho) \tag{22}$$

and the solution is $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$. Clearly $\omega^2 > 0$ to have an oscillation otherwise ω is imaginary and the solution blows up.

So to have a stable orbit:

$$\frac{3}{\rho} + \frac{g'(\rho)}{g(\rho)} > 0 \tag{23}$$

Note that if $F(r) = -K/r^n$, this condition is equivalent to $n < 3$, as we saw earlier.