Lecture 19
Monday - October 17, 2005
Written or last updated: October 17, 2005

P441 – Analytical Mechanics - I
Gravitational Field and its Potential

Isaac Newton

What Isaac Newton achieved was truly remarkable and he was also a remarkable man. To abstract from all the astronomical observation known at the time – the motion of the moon and planets – down to a simple universal law that explains all terrestrial and astronomical phenomena involving masses is awesome. On top of that he invented calculus so that he could have the mathematics needed to do calculations. James Gleick recently wrote a nice and short biography of Newton. He also wrote a biography of Feynman. The definitive biography of Newton was written by the late Richard Westfall, who was a professor at Indiana University.

http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Newton.html

Figure 1: Sir Isaac Newton - on a medallion issued by the Royal Mint upon his death.

Gravitational field

We start with the following assumption about the gravitational force exerted by a particle of mass m fixed at the origin of coordinates on a test mass m_0 located a distance r from the origin. That force is:

$$\mathbf{F} = -G\frac{mm_0}{r^2}\mathbf{e}_R \tag{1}$$

This assumption comes from observation – it is not derived. The minus sign tells us that the force is attactive. Also the force acts along the line connecting the two particles and varies inversely as the square of the distance. The gravitational constant G is a universal constant. As far as we know, it has the same value throughout the universe and that value is $G = 6.672 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}$.

In analogy to how we define the electric field, we can forget about making reference to the test mass m_0 and define a vector field given by:

$$\mathbf{g} = -G\frac{m}{r^2}\mathbf{e}_R \tag{2}$$

We can imagine the space around the mass m being filled with a vector field $\mathbf{g}$ and a particle of mass m_0 placed in this field will experience a force $\mathbf{F} = m_0\mathbf{g}$.

From equation 2 it immediately follows that $\vec{\nabla} \times \mathbf{g} = 0$ which means the the field $\mathbf{g}$ is conservative and we can associate a potential Φ with this field where $\mathbf{g} = -\vec{\nabla}\Phi$ – in analogy to how we associate an electrical potential ϕ with the electric field $\mathbf{E}$.

Superposition and the gravitational field Here is something else that follows from observation. The gravitational field follows the principle of superposition. If m_0 is in the gravitational field of two other masses m_1 and m_2 then we can compute the vector force of m_1 on m_0 ignoring m_2 and then compute the force of m_2 on m_0 ignoring m_1 and then vectorially add the two. It did not have to be this way – that's how the gravitational force works and being able to use superposition makes life a whole lot easier.

In general, the field at a point located by $\mathbf{r}_0$ due to N masses m_i located at $\mathbf{r}_i$ is given by:

$$\mathbf{g} = -G\sum_{i=1}^{N}\frac{m_i}{r_i^2}\mathbf{e}_{0i} \tag{3}$$

where $\mathbf{e}_{0i}$ is a unit vector along $\mathbf{r}_i - \mathbf{r}_0$.

Gravitational field and potential

The gravitational potential a distance r away from a particle of mass m is given by:

$$\Phi(r) = -\frac{Gm}{r} \tag{4}$$

The potential at some point due to N masses m_i each located at $\mathbf{r}_i$ from the point is given by:

$$\Phi = -G\sum_{i=1}^{N}\frac{m_i}{r_i} \tag{5}$$

Gravitational potential of a thin mass shell

Figure 2 shows a thin spherical shell of mass M and radius a. We want to calculate the gravitational potential Φ at point P a distance R from the center of the shell. We assume that the mass is uniformly distributed over the surface of the shell and the mass per area is $\sigma = M/4\pi R^2$.

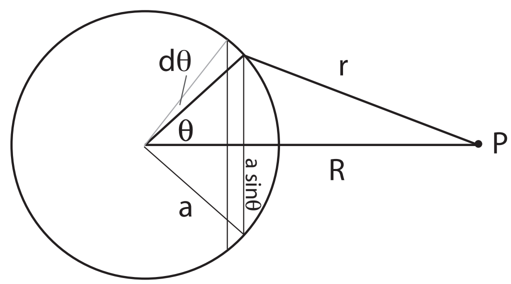

Figure 2: A thin spherical shell of mass M produces a gravitational potential as point P.

We will divide the shell into little strips as, shown in the figure, each of which goes around the shell. The area of the strip is $dA = 2\pi a^2 \sin\theta d\theta$ and the mass of this strip is $dM = \sigma dA$ and all points of this strip are equidistant from the point P and this distance is r. The contribution to the potential from this strip is:

$$d\Phi(r) = -\frac{GdM}{r} \tag{6}$$

We have an expression for the square of the length of r: $r^2 = a^2 + R^2 - 2aR\cos\theta$. As we integrate over all the strips on the shell note that r varies while a and R are constant – but θ changes. Taking the differential of r^2 results in $2rdr = 2aR\sin\theta d\theta$. We now have:

$$d\Phi(r) = -\frac{\sigma G 2\pi a^2 \sin\theta d\theta}{r} = -\frac{\sigma G 2\pi a dr}{R} \tag{7}$$

When we integrate over the shell r goes from $R - a$ to $R + a$:

$$\Phi(r) = -\frac{\sigma G 2\pi a}{R}\int_{R-a}^{R+a} dr = -\frac{GM}{R} \tag{8}$$

The shell of mass produces a potential at point P as if all the mass of the shell were concentrated at the center of the shell. Now suppose that the point P is inside the shell. What changes? The limits of the integration variable r are now from $a - R$ to $a + R$:

$$\Phi(r) = -\frac{\sigma G 2\pi a}{R}\int_{a-R}^{a+R} dr = -\frac{GM}{a} \tag{9}$$

The significance of this is that the potential is the same everywhere inside the shell.

The gravitational vector field $\mathbf{g}$ is related to the potential by the relation $\vec{g} = -\vec{\nabla}\Phi$. So we see that everywhere inside the shell $\mathbf{g} = 0$ and outside $\mathbf{g} = -GM\hat{e}_r/r$.

Gravitational potential of a thick mass shell

Consider now a shell with finite thickness with inner radius b and outer radius a. We assume the shell has mass M that is uniformly distributed with constant mass per volume $\rho = M/V$ where $V = 4\pi(a^3 - b^3)/3$.

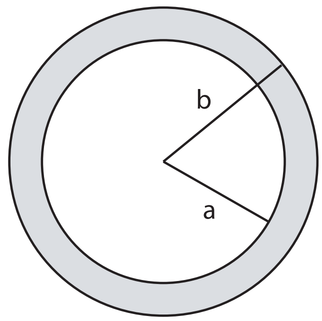

Figure 3: A thick spherical shell of mass M with inner radius a and outer radius b.

The potential outside the shell, a distance R from the center of the shell where $R > a$ is simply $\Phi(R) = -GM/R$. What about inside the shell where $R < b$? We know that the potential everywhere inside is the same and equal to the value of the potential at the center. So we calculate the potential here. Consider a shell of thickness dr and radius r where $b < r < a$. The volume of this shell is $dV = 4\pi r^2 dr$ and the mass is $dm = \rho dV$ and contributes potential $d\Phi = -Gdm/r$. Integrating:

$$\Phi = -G\rho 4\pi \int_a^b \frac{r^2 dr}{r} = -G\rho 2\pi(a^2 - b^2) \tag{10}$$

Now what about a point R where $b < R < a$? The mass contained from $r = b$ to $r = R$ is $M_1 = 4\pi(R^3 - b^3)/3$ and contributes potential $\Phi_1 = -GM_1/R$ while the mass from $r = R$ to $r = a$ contributes $\Phi_2 = -G\rho 2\pi(a^2 - R^2)$. The total potential:

$$\Phi(R) = -\frac{G\rho 4\pi}{3R}(R^3 - b^3) - G\rho 2\pi(a^2 - R^2) \tag{11}$$

Flux of the gravitational field

Suppose we have an imaginary sphere of radius R centered on a point mass m. At each point on the surface of the sphere the field vector $\mathbf{g}$ is perpendicular to the surface, directed towards the center of the sphere and everywhere on the surface has magnitude $g = GmR^2/$. The flux of $\mathbf{g}$ over the sphere is $-g4\pi R^2$ or flux $= -4\pi Gm$. The minus sign occurs because at any point on the surface of the sphere the normal to the surface points away from the center of the sphere and $\mathbf{g}$ is anti-parallel to the surface normal. The flux for this special case is independent of the radius of the sphere.

Now suppose the particle of mass m is not centered in the sphere? Before we answer that question please refer to Figure 4. We draw two lines through mass m and then a line (dashed line) in between. Rotate the original two lines through 180° about the dashed line to sweep out two cones. Consider where these cones intercept some surface. First look at the intercepted and shaded areas I and III. Both areas are perpendicular to the dotted line (by assumption). We'll assume that these are small enough so that we can assume that all parts of the areas are perpendicular to the dotted line. The distances these area are from

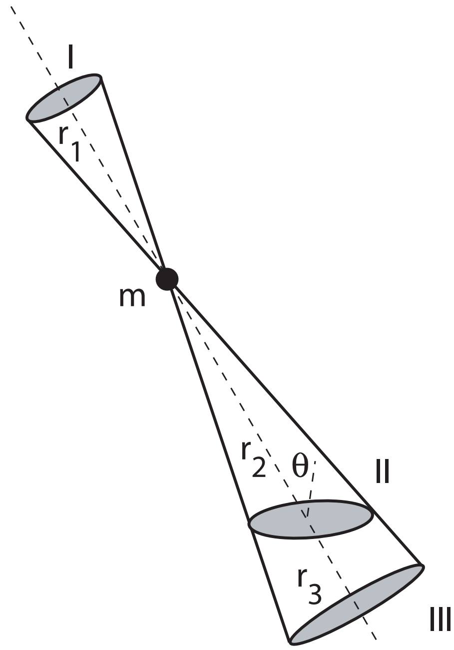

Figure 4: A thick spherical shell of mass M with inner radius a and outer radius b.

the mass are r_1 and r_3. Area III is larger than area I in the ratio r_3^2/r_1^2. But the magnitude of $\mathbf{g}$ at area III is smaller than the magnitude over area I by the ratio r_1^2/r_3^2. For both areas $\mathbf{g}$ is perpendicular to the surface. So the flux of $\mathbf{g}$ over the two areas is the same.

Now consider area II. It's surface normal makes angle θ with respect to the dotted line. Area II is larger than area I r_2^2/r_1^2 but there is yet another factor $1/\cot\theta$. The magnitude of $\mathbf{g}$ at area II is smaller than the magnitude over area I by the ratio r_1^2/r_2^2 and in addition, when we calculate the flux we need to include $\cos\theta$ to account for the angle between the surface normal and $\mathbf{g}$. The upshot of this all is that the flux over all shaded three areas is the same and equal to any other area intercepted by these cones. Using this, we can see after adding up over all cones, that the flux of $\mathbf{g}$ over a sphere is the same no matter where the mass is located inside the sphere. The the surface does not even have to be a sphere. It could be any closed surface – you still get the same answer as long as the mass is inside.

So with complete generality you can write:

$$\oint \mathbf{g} \cdot \hat{\mathbf{n}} da = -4\pi G m \tag{12}$$

The integral symbol $\oint$ indicates a closed surface. Using our cone argument we can also show that the flux over a closed surface is zero if the mass is outside the surface. Finally, using superposition, when mass is continuously distributed with volume charge density ρ that might vary over space then:

$$\oint \mathbf{g} \cdot \hat{\mathbf{n}} da = -4\pi G \int \rho dV \tag{13}$$

The volume integral is taken over the volume bounded by the closed surface over which the flux (LHS) is computed.