P441 – Analytical Mechanics - I
Examples in Lagrangian Mechanics

Sample problems using Lagrangian mechanics

Here are some sample problems. I will assign similar problems for the next problem set.

Example 1 In Figure 1 we show a box of mass m sliding down a ramp of mass M. The ramp moves without friction on the horizontal plane and is located by coordinate x_1. The box also slides without friction on the ramp and is located by coordinate x_2 with respect to the ramp.

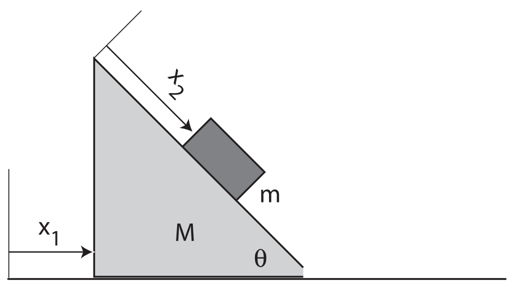

Figure 1: A box slides down a ramp without friction and the ramp slides along a horizontal surface without friction.

The kinetic energy of the ramp T_M is given by:

$$T_M = \frac{1}{2} M \dot{x}_1^2 \tag{1}$$

and the kinetic energy of the box T_m is given by:

$$T_m = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1 \dot{x}_1 \cos\theta \right) \tag{2}$$

The velocity of the box is obviously derived from the vector sum of the velocity relative to the ramp and the velocity of the ramp.

The potential energy of the system is just the potential energy of the box and that leads to:

$$U = -mgx_2 \sin\theta \tag{3}$$

and finally the Lagrangian is given by:

$$L(x_1, \dot{x}_1, x_2, \dot{x}_2) = T_M + T_m - U = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\dot{x}_1\cos\theta\right) + mgx_2\sin\theta \tag{4}$$

The equations of motion are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial L}{\partial x_1} \tag{5}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{\partial L}{\partial x_2} \tag{6}$$

Equation 5 leads to:

$$\frac{d}{dt}\left[m(\dot{x}_1 + \dot{x}_2\cos\theta) + M\dot{x}_1\right] = 0 \tag{7}$$

The RHS of equation 7 is zero because the Lagrangian does not explicitly depend on x_1. The quantity in the brackets is the total momentum in the horizontal direction which is a constant since there are no forces on the system in this direction.

Application of equation 6 leads to:

$$\frac{d}{dt}(\dot{x}_2 + \dot{x}_1\cos\theta) = g\sin\theta \tag{8}$$

Carrying out the time derivatives in equations 7 and 8 leads to two linear equations in $\ddot{x}_1$ and $\ddot{x}_2$. These two equations in two unknowns can be solved to yield:

$$\ddot{x}_1 = \frac{-g\sin\theta\cos\theta}{(m+M)/m - \cos^2\theta} \tag{9}$$

and

$$\ddot{x}_2 = \frac{g\sin\theta}{1 - m\cos^2\theta/(m+M)} \tag{10}$$

The RHS of both of the above equations are constants – we can easily find $x_1(t)$ and $x_2(t)$.

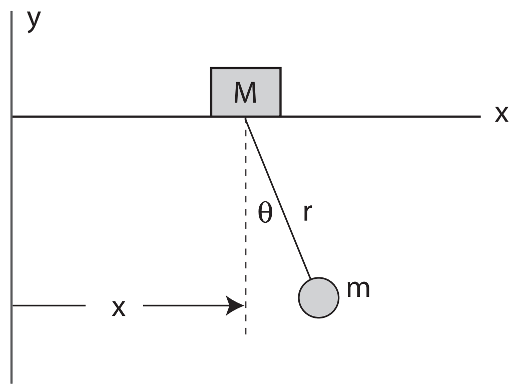

Figure 2: A simple pendulum attached to a support that is free to move.

Example 2 Figure 2 shows a simple pendulum consisting of a string of length r and a bob of mass m that is attached to a support of mass M. The support moves without friction on the horizontal plane.

The x component of the velocity of the bob is given by $\dot{x} + r\dot{\theta}\cos\theta$ and the y component by $r\dot{\theta}\sin\theta$. So the overall kinetic energy of the system is given by:

$$T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\left[\dot{x}^2 + r^2\dot{\theta}^2 + 2\dot{x}r\dot{\theta}\cos\theta\right] \tag{11}$$

The potential energy is:

$$U = -mgr\cos\theta \tag{12}$$

and the Lagragian:

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2}(m + M)\dot{x}^2 + \frac{1}{2}(r^2\dot{\theta}^2 + 2\dot{x}r\dot{\theta}\cos\theta) + mgr\cos\theta \tag{13}$$

Apply Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{14}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \tag{15}$$

to get:

$$\frac{d}{dt}\Big[(m+M)\dot{x} + mr\dot{\theta}\cos\theta\Big] = 0 \tag{16}$$

and

$$\frac{d}{dt}\Big[m(r^2\dot{\theta} + \dot{x}r\cos\theta)\Big] = -m(\dot{x}r\dot{\theta} + gr)\sin\theta \tag{17}$$

Equatio 16 leads to a conserved quantity – the momentum along the x direction and equation 17 gives us:

$$\ddot{\theta} + \frac{\ddot{x}}{r}\cos\theta + \frac{g}{r}\sin\theta = 0 \tag{18}$$

Note that when the support is moving with constant velocity ($\ddot{x} = 0$) we just have the equation of motion for a pendulum.

Suppose we set $\ddot{\theta} = 0$. Then:

$$\tan\theta = -\frac{\ddot{x}}{g} \tag{19}$$

If we accelerate the support to the right then the pendulum hangs motionless at the angle given by the above equation.

Example 3 The figure shows a mass M connected to another mass m. Mass M moves without friction along a circle of radius r on the horizontal surface of a table. The two masses are connected by a massless string of length ℓ that passes through a hole in the table. At given time the mass M is located by r and θ.

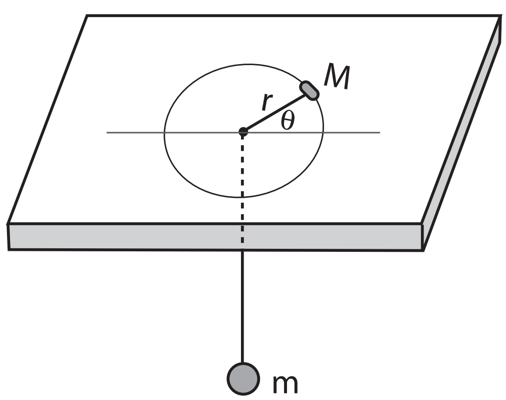

Figure 3: Example 3.

The kinetic energies of the mass on the table T_M and the hanging mass T_m are given by:

$$T_M = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$T_m = \frac{1}{2}m\dot{r}^2$$

The potential energy is $U = mgr$ so the Lagrangian is:

$$L = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}m\dot{r}^2 - mgr \tag{20}$$

Applying:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

The equations of motion are:

$$(M + m)\ddot{r} = Mr\dot{\theta}^2 - mg$$
$$\frac{d}{dt}\left(Mr^2\dot{\theta}\right) = 0$$

The quantity in the parenthesis in the second of the above two equations is a conserved quantity since the Lagrangian does not explicitly depend on θ.

Example 4 A particle of mass m is free to move without friction on the inside of a hemispherical bowl whose axis is aligned along the vertical. The radius of the hemisphere is R and the particle is located by the polar angle θ and the azimuthal angle ϕ.

The Lagrangian is:

$$L = \frac{mR^2}{2}\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + mgR\cos\theta \tag{21}$$

Applying:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}$$

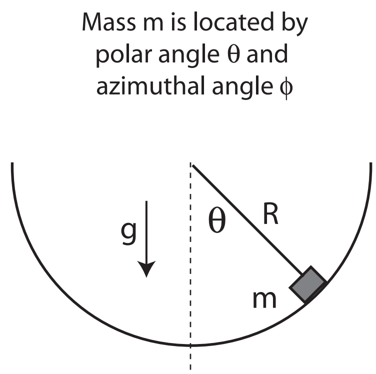

Figure 4: Example 4.

the equations of motion become:

$$
\begin{aligned}
mR^2\ddot{\theta} &= -mgR\sin\theta + mR^2\sin\theta\cos\theta\dot{\phi}^2 \\
\frac{d}{dt}\left(mR^2\sin^2\theta\dot{\phi}\right) &= 0
\end{aligned}
$$

If $\dot{\phi} = 0$ then the first of these looks like the equation of motion for a simple pendulum: $\ddot{\theta} = -(g/R)\sin\theta$ and the quantity in the parenthesis in the second equation is a constant of the motion, a conserved quantity, because the Lagrangian does not explicitly depend on ϕ.