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## **THE FIELDS OF MOVING CHARGES**

### FROM OERSTED TO EINSTEIN

**5.1** In the winter of 1819–1820 Hans Christian Oersted was lecturing on electricity, galvanism, and magnetism to advanced students at the University of Copenhagen. *Electricity* meant electrostatics; *galvanism* referred to the effects produced by continuous currents from batteries, a subject opened up by Galvani's chance discovery and the subsequent experiments of Volta; *magnetism* dealt with the already ancient lore of lodestones, compass needles, and the terrestrial magnetic field. It seemed clear to some that there must be a relation between galvanic currents and electric charge, although there was little more direct evidence than the fact that both could cause shocks. On the other hand, magnetism and electricity appeared to have nothing whatever to do with one another. Still Oersted had a notion, vague perhaps, but tenaciously pursued, that magnetism like the galvanic current might be a sort of "hidden form" of electricity. Groping for some manifestation of this, he tried before his class the experiment of passing a galvanic current through a wire which ran above and at right angles to a compass needle. It had no effect. After the lecture, something impelled him to try the experiment with a wire running parallel to the compass needle. The needle swung wide—and when the galvanic current was reversed it swung the other way!

The scientific world was more than ready for this revelation. A ferment of experimentation and discovery followed as soon as the word reached other laboratories. Before long Ampère, Faraday, and others had worked out an essentially complete and exact description of the magnetic action of electric currents. Faraday's crowning discovery of electromagnetic induction came less than 12 years after Oersted's experiment. In the previous two centuries since the publication in 1600 of William Gilbert's great work *De Magnete*, man's understanding of magnetism had advanced not at all. Out of these experimental discoveries there grew the complete classical theory of electromagnetism. Formulated mathematically by Maxwell, it was triumphantly corroborated by Hertz's demonstration of electromagnetic waves in 1888.

Special relativity has its historical roots in electromagnetism. Lorentz, exploring the electrodynamics of moving charges, was led very close to the final formulation of Einstein. And Einstein's great paper of 1905 was entitled not "Theory of Relativity," but rather "On the Electrodynamics of Moving Bodies." Today we see in the postulates of relativity and their implications a wide framework, one that embraces all physical laws and not solely those of electromagnetism. We expect any complete physical theory to be relativistically invariant. It ought to tell the same story in all inertial frames of reference. As it happened, physics already *had* one relativistically invariant theory—Maxwell's electromagnetic theory—long before the significance of relativistic invariance was recognized. Whether the ideas of special relativity could have evolved in the absence of a complete theory of

the electromagnetic field is a question for the historian of science to speculate about; probably it can't be answered. We can only say that the actual history shows rather plainly a path running from Oersted's compass needle to Einstein's postulates.

Still, relativity is not a branch of electromagnetism, nor a consequence of the existence of light. The central postulate of special relativity, which no observation has yet contradicted, is the equivalence of reference frames moving with constant velocity with respect to one another. Indeed, it is possible, without even mentioning light, to derive the formulas of special relativity from nothing more than that postulate and the assumption that all spatial directions are equivalent.\* The universal constant  $c$  then appears in these formulas as a limiting velocity, approached by an energetic particle but never exceeded. Its value can be ascertained by an experiment that does not involve light or anything else (such as neutrinos) which are believed to travel at precisely that speed. In other words, we would have special relativity even if electromagnetic waves could not exist.

Later in this chapter we are going to follow the historical path from Oersted to Einstein almost in reverse. We'll take special relativity as given, and ask how an electrostatic system of charges and fields looks in another reference frame. In this way we shall find the forces that act on electric charges in motion, including the force that acts between electric currents. Magnetism, seen from this viewpoint, is a relativistic aspect of electricity.† But first, let's review some of the phenomena we shall be trying to explain.

## MAGNETIC FORCES

**5.2** Two wires running parallel to one another and carrying currents in the same direction are drawn together. The force on one of the wires, per unit length of wire, proves to be proportional to the dis-

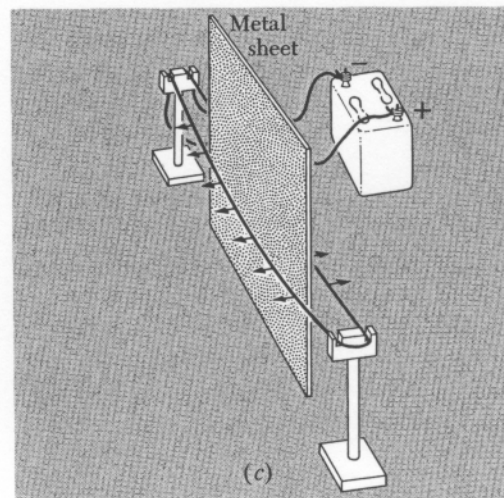
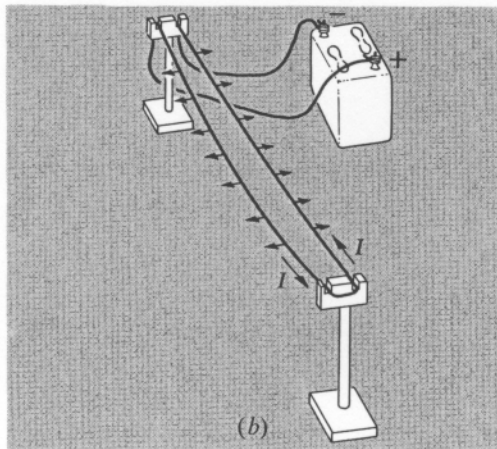
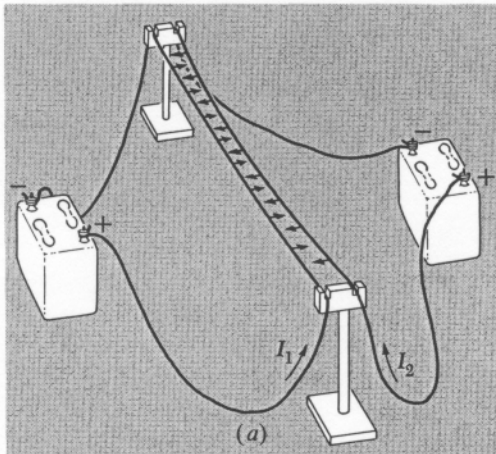
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\*See N. David Mermin, "Relativity Without Light," *American Journal of Physics*, 52:119 (1984), in which it is shown that the most general law for the addition of velocities which is consistent with the equivalence of inertial frames must have the form  $v = (v_1 + v_2)/(1 + v_1v_2/c^2)$ , identical to our Eq. 6 in Appendix A. To discover the value of the constant  $c$  in our universe we need only measure with adequate accuracy three lower speeds  $v$ ,  $v_1$ , and  $v_2$ . Reference to other derivations of special relativity from the postulate of frame equivalence will be found in Mermin's article.

†The earliest exposition of this approach, to my knowledge, is the article by L. Page, A Derivation of the Fundamental Relations of Electrodynamics from Those of Electrostatics, *American Journal of Science*, XXXIV: 57 (1912). It was natural for Page, writing only 7 years after Einstein's revolutionary paper, to consider relativity more in need of confirmation than electrodynamics. His concluding sentence reads: "Viewed from another standpoint, the fact that we have been able, by means of the principle of relativity, to deduce the fundamental relations of electrodynamics from those of electrostatics, may be considered as some confirmation of the principle of relativity."

**FIGURE 5.1**

(a) Parallel wires carrying currents in the same direction are pulled together. (b) Parallel wires carrying currents in opposite directions are pushed apart. (c) These forces are not affected by putting a metal plate between the wires.



tance between the wires (Fig. 5.1a). Reversing the direction of one of the currents changes the force to one of repulsion. Thus the two sections of wire in Fig. 5.1b, which are part of the same circuit, tend to fly apart. There is some sort of “action at a distance” between the two filaments of steady electric current. It seems to have nothing to do with any static electric charge on the surface of the wire. There may be some such charge and the wires may be at different potentials, but the force we are concerned with depends only on the charge *movement* in the wires, that is, on the two currents. You can put a sheet of metal between the two wires without affecting this force at all (Fig. 5.1c). These new forces that come into play when charges are moving are called *magnetic*.

Oersted’s compass needle (Fig. 5.2a) doesn’t look much like a direct-current circuit. We now know, however, as Ampère was the first to suspect, that magnetized iron is full of perpetually moving charges—electric currents on an atomic scale. A slender coil of wire with a battery to drive current through it (Fig. 5.2b) behaves just like the compass needle under the influence of a nearby current.

Observing the motion of a free charged particle, instead of a wire carrying current, we find the same thing happening. In a cathode ray tube, electrons that would otherwise follow a straight path are deflected toward or away from an external current-carrying wire, depending on the relative direction of the current in that wire (Fig. 5.3). You are already familiar with this from the laboratory, and you know that this interaction of currents and other moving charges can

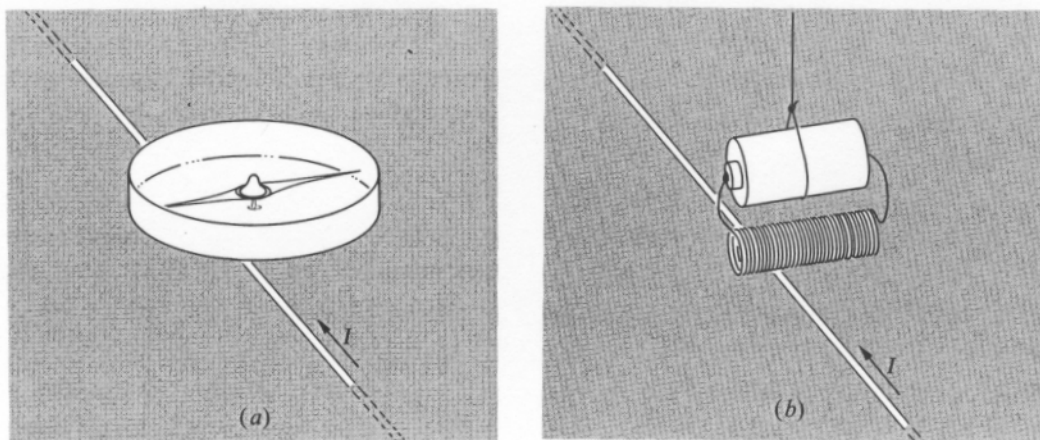
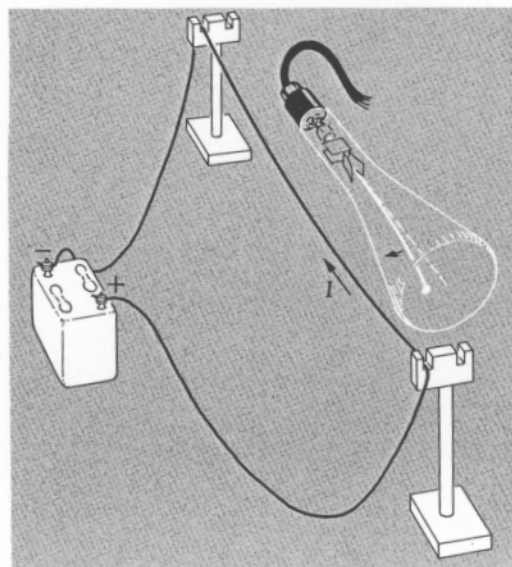


FIGURE 5.2

A compass needle (a) and a coil of wire carrying current (b) are similarly influenced by current in a nearby conductor. The direction of the current  $I$  is understood to be that in which positive ions would be moving if they were the carriers of the current. In the earth's magnetic field the black end of the compass needle would point north.

FIGURE 5.3

An example of the attraction of currents in the same direction. Compare with Fig. 5.1a. We can also describe it as the deflection of an electron beam by a magnetic field.



be described by introducing a *magnetic field*. (The electric field, remember, was simply a way of describing the action at a distance between stationary charges that is expressed in Coulomb's law.) We say that an electric current has associated with it a magnetic field which pervades the surrounding space. Some other current, or any moving charged particle which finds itself in this field, experiences a force proportional to the strength of the magnetic field in that locality. The force is always perpendicular to the velocity, for a charged particle. The entire force on a particle carrying charge  $q$  is given by

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c} \mathbf{v} \times \mathbf{B} \quad (1)$$

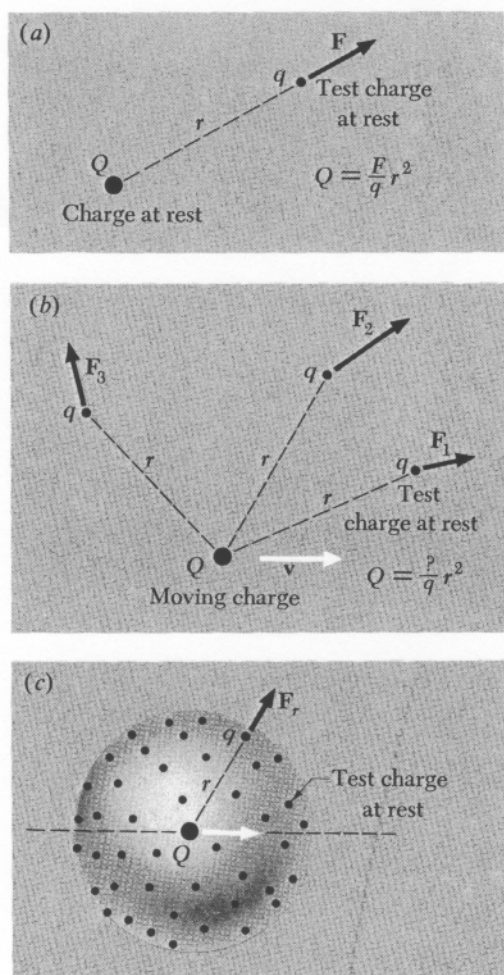
where  $\mathbf{B}$  is the magnetic field.†

We shall take Eq. 1 as the definition of  $\mathbf{B}$ . The inclusion of a factor  $1/c$  in the second term appears, at this stage, quite arbitrary. We are free to include it since we have not previously specified the units for  $\mathbf{B}$ . We shall deal with the question of units at the beginning of the next chapter. All that concerns us now is that the magnetic field strength is a vector which determines the velocity-proportional part of the force on a moving charge. In other words, the command, "Mea-

†Here for the first time we make use of the vector product, or *cross product*, of two vectors. A reminder: The vector  $\mathbf{v} \times \mathbf{B}$  is a vector perpendicular to both  $\mathbf{v}$  and  $\mathbf{B}$  and of magnitude  $vB \sin \theta$ , where  $\theta$  is the angle between the directions of  $\mathbf{v}$  and  $\mathbf{B}$ . A right-hand rule determines the sense of the direction of the vector  $\mathbf{v} \times \mathbf{B}$ . In our Cartesian coordinates  $\hat{x} \times \hat{y} = \hat{z}$  and  $\mathbf{v} \times \mathbf{B} = \hat{x}(v_y B_z - v_z B_y) + \hat{y}(v_z B_x - v_x B_z) + \hat{z}(v_x B_y - v_y B_x)$ .

**FIGURE 5.4**

(a) The magnitude of a charge at rest is determined by the force on a test charge at rest and Coulomb's law. (b) In the case of a moving charge, the force, for all we know now, may depend on the position of the test charge. If so, we can't use procedure (a). (c) At the instant  $Q$  passes through the center of the spherical array of test charges, measure the radial force component on each, and use the average value of  $F_r$  to determine  $Q$ . This is equivalent to measuring the surface integral of  $\mathbf{E}$ .



sure the direction and magnitude of the vector  $\mathbf{B}$  at such and such a place," calls for the following operations: Take a particle of known charge  $q$ . Measure the force on  $q$  at rest, to determine  $\mathbf{E}$ . Then measure the force on the particle when its velocity is  $\mathbf{v}$ ; repeat with  $\mathbf{v}$  in some other direction. Now find a  $\mathbf{B}$  that will make Eq. 1 fit all these results; that is the magnetic field at the place in question.

Clearly this doesn't *explain* anything. Why does Eq. 1 work? Why can we always find a  $\mathbf{B}$  that is consistent with this simple relation, for all possible velocities? We want to understand why there is a velocity-proportional force. It is really most remarkable that this force is strictly proportional to  $v$ , and that the effect of the electric field does not depend on  $v$  at all! In the following pages we'll see how this comes about. It will turn out that a field  $\mathbf{B}$  with these properties *must* exist if the forces between electric charges obey the postulates of special relativity. Seen from this point of view, magnetic forces are a relativistic aspect of charge in motion.

A review of the essential ideas and formulas of special relativity is provided in Appendix A. This would be a good time to read through it.

### MEASUREMENT OF CHARGE IN MOTION

**5.3** How are we going to measure the quantity of electric charge on a moving particle? Until this question is settled, it is pointless to ask what effect motion has on charge itself. A charge can only be measured by the effects it produces. A point charge  $Q$  which is at rest can be measured by determining the force that acts on a test charge  $q$  a certain distance away (Fig. 5.4a). That is based on Coulomb's law. But if the charge we want to measure is moving, we are on uncertain ground. There is now a special direction in space, the instantaneous direction of motion. It could be that the force on the test charge  $q$  depends on the *direction* from  $Q$  to  $q$ , as well as on the distance between the two charges. For different positions of the test charge, as in Fig. 5.4b, we would observe different forces. Putting these into Coulomb's law would lead to different values for the same quantity  $Q$ . Also we have as yet no assurance that the force will always be in the direction of the radius vector  $\mathbf{r}$ .

To allow for this possibility, let's agree to define  $Q$  by averaging over all directions. Imagine a large number of infinitesimal test charges distributed evenly over a sphere (Fig. 5.4c). At the instant the moving charge passes the center of the sphere, the radial component of force on each test charge is measured, and the average of these force magnitudes is used to compute  $Q$ . Now this is just the operation that would be needed to determine the surface integral of the electric field over that sphere, at time  $t$ . The test charges here are all at rest,

remember; the force on  $q$  per unit charge gives, by definition, the electric field at that point. This suggests that Gauss's law, rather than Coulomb's law, offers the natural way† to define quantity of charge for a moving charged particle, or for a collection of moving charges. We can frame such a definition as follows.

The amount of electric charge in a region is defined by the surface integral of the electric field  $\mathbf{E}$  over a surface  $S$  enclosing the region. This surface  $S$  is fixed in some coordinate frame  $F$ . The field  $\mathbf{E}$  is measured, at any point  $(x, y, z)$  and at time  $t$  in  $F$ , by the force on a test charge *at rest* in  $F$ , at that time and place. The surface integral is to be determined for a particular time  $t$ . That is, the field values used are those measured simultaneously by observers deployed all over  $S$ . (This presents no difficulty, for  $S$  is stationary in the frame  $F$ .) Let us denote such a surface integral, over  $S$  at time  $t$ , by

$$\int_{S(t)} \mathbf{E} \cdot d\mathbf{a} \quad (2)$$

We define the amount of charge inside  $S$  as  $1/4\pi$  times this integral:

$$Q = \frac{1}{4\pi} \int_{S(t)} \mathbf{E} \cdot d\mathbf{a} \quad (3)$$

It would be embarrassing if the value of  $Q$  so determined depended on the size and shape of the surface  $S$ . For a stationary charge it doesn't—that is Gauss's law. But how do we know that Gauss's law holds when charges are moving? Fortunately it does. We can take that as an experimental fact. This fundamental property of the electric field of moving charges permits us to define quantity of charge by Eq. 3. From now on we can speak of the amount of charge in a region or on a particle, and that will have a perfectly definite meaning even if the charge is in motion.

Figure 5.5 summarizes these points in an example. Two protons and two electrons are shown in motion, at a particular instant of time. It is a fact that the surface integral of the electric field  $\mathbf{E}$  over the surface  $S_1$  is precisely equal to the surface integral over  $S_2$  evaluated at the same instant, and we may use this integral, as we always have used Gauss's law in electrostatics, to determine the total charge enclosed. Figure 5.6 raises a new question. What if the same particles had some other velocities? For instance, suppose the two protons and two electrons combine to form a hydrogen molecule. Will the total charge appear exactly the same as before?

†It is not the only *possible* way. You could, for instance, adopt the arbitrary rule that the test charge must always be placed directly ahead (in the direction of motion) of the charge to be measured. Charge so defined would *not* have the simple properties we are about to discuss, and your theory would prove clumsy and complicated.

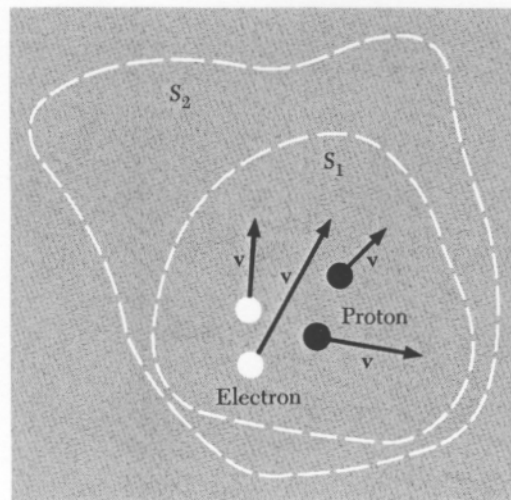
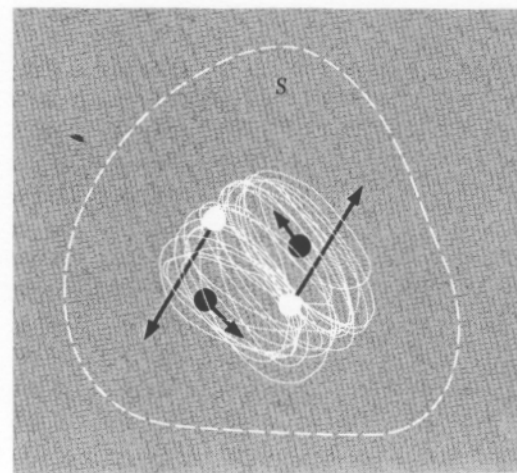


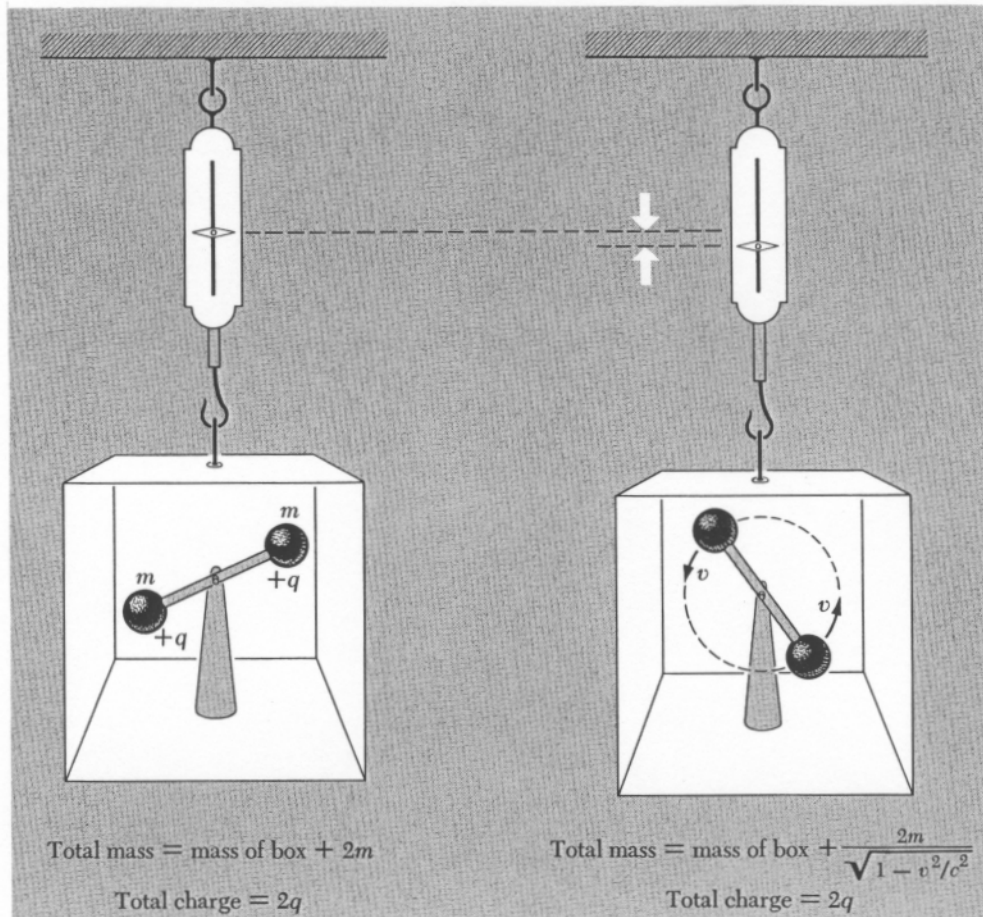
FIGURE 5.5

Gauss' law remains valid for the field of moving charges. The flux of  $\mathbf{E}$  through  $S_2$  is equal to the flux of  $\mathbf{E}$  through  $S_1$ , evaluated at the same instant of time.

FIGURE 5.6

Does the flux of  $\mathbf{E}$  through  $S$  depend on the state of motion of the charged particles? Is the surface integral of  $\mathbf{E}$  over  $S$  the same as in Fig. 5.5? Here the particles are bound together as a hydrogen molecule.



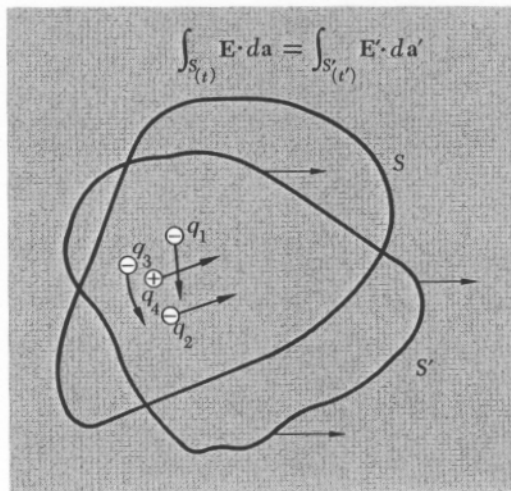
**FIGURE 5.7**

An imaginary experiment to show the invariance of charge. The charge in the box is to be measured by measuring the electric field all around the box or, equivalently, by measuring the force on a distant test charge.

around with very different speeds. The difference in energy shows up as a measurable difference in mass. There is no detectable difference, to very high precision, in the electric charge of the two ions.

This invariance of charge lends a special significance to the fact of charge quantization. We emphasized in Chapter 1 the importance—and the mystery—of the fact that every elementary charged particle has a charge equal in magnitude to that of every other such particle. We now observe that this precise equality holds not only for two particles at rest with respect to one another, but for *any* state of relative motion.

The experiments we have described, and many others, show that the value of our Gauss's law surface integral  $\int_S \mathbf{E} \cdot d\mathbf{a}$  depends only on the number and variety of charged particles inside  $S$ , and not on



**FIGURE 5.8**

The surface integral of  $\mathbf{E}$  over  $S$  is equal to the integral of  $\mathbf{E}'$  over  $S'$ . The charge is the same in all frames of reference.

how they are moving. According to the postulate of relativity, such a statement must be true for *any* inertial frame of reference if it is true for one. Therefore if  $F'$  is some *other* inertial frame, moving with respect to  $F$ , and if  $S'$  is a closed surface in *that* frame which at time  $t'$  encloses the same charged bodies that were enclosed by  $S$  at time  $t$ , we must have

$$\int_{S(t)} \mathbf{E} \cdot d\mathbf{a} = \int_{S'(t')} \mathbf{E}' \cdot d\mathbf{a}' \quad (4)$$

The field  $\mathbf{E}'$  is of course measured in  $F'$ , that is, it is defined by the force on a test charge at rest in  $F'$ . The distinction between  $t$  and  $t'$  must not be overlooked. As we know, events that are simultaneous in  $F$  need not be simultaneous in  $F'$ . Each of the surface integrals in Eq. 4 is to be evaluated at one instant in *its* frame. If charges lie on the boundary of  $S$ , or of  $S'$ , one has to be rather careful about ascertaining that the charges within  $S$  at  $t$  are the same as those within  $S'$  at  $t'$ . If the charges are well away from the boundary, as in Fig. 5.8 which is intended to illustrate the relation in Eq. 4, there is no problem in this respect.

Equation 4 is a formal statement of the relativistic invariance of charge. We can choose our gaussian surface in *any* inertial frame; the surface integral will give a number independent of the frame. Invariance of charge is not the same as charge conservation, which was discussed in Chapter 4 and is expressed mathematically in the equation

$$\text{div } \mathbf{J} = \frac{-\partial \rho}{\partial t}$$

Charge *conservation* implies that, if we take a closed surface fixed in some coordinate system and containing some charged matter, and if no particles cross the boundary, then the total charge inside that surface remains constant. Charge *invariance* implies that, if we look at this collection of stuff from any other frame of reference, we will measure exactly the same amount of charge. Energy is conserved, but energy *is not* a relativistic invariant. Charge is conserved, and charge *is* a relativistic invariant. In the language of relativity theory, energy is one component of a four-vector, while charge is a scalar, an invariant number, with respect to the Lorentz transformation. This is an observed fact with far-reaching implications. It completely determines the nature of the field of moving charges.

### ELECTRIC FIELD MEASURED IN DIFFERENT FRAMES OF REFERENCE

**5.5** If charge is to be invariant under a Lorentz transformation, the electric field  $\mathbf{E}$  has to transform in a particular way. "Transforming

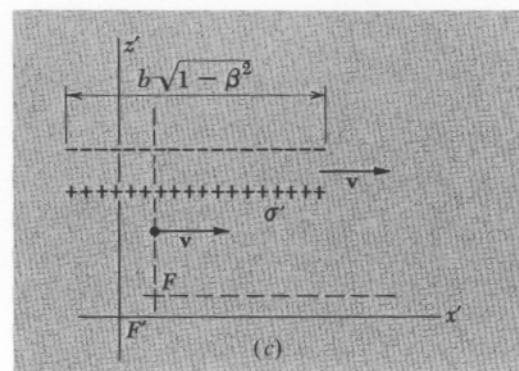
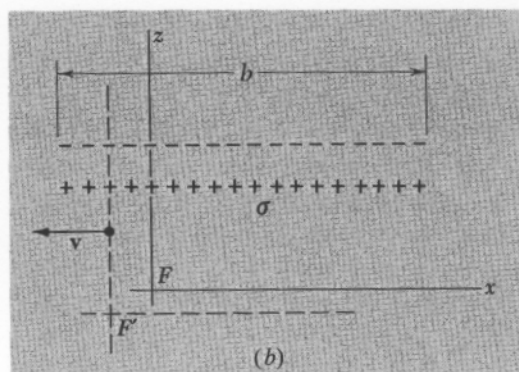
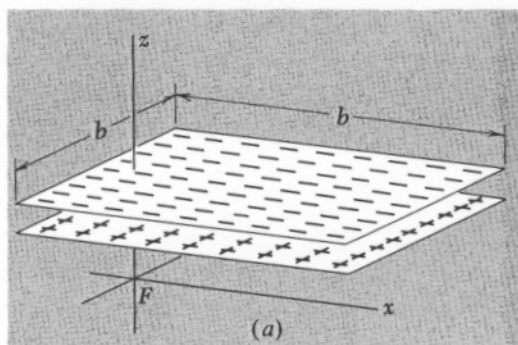


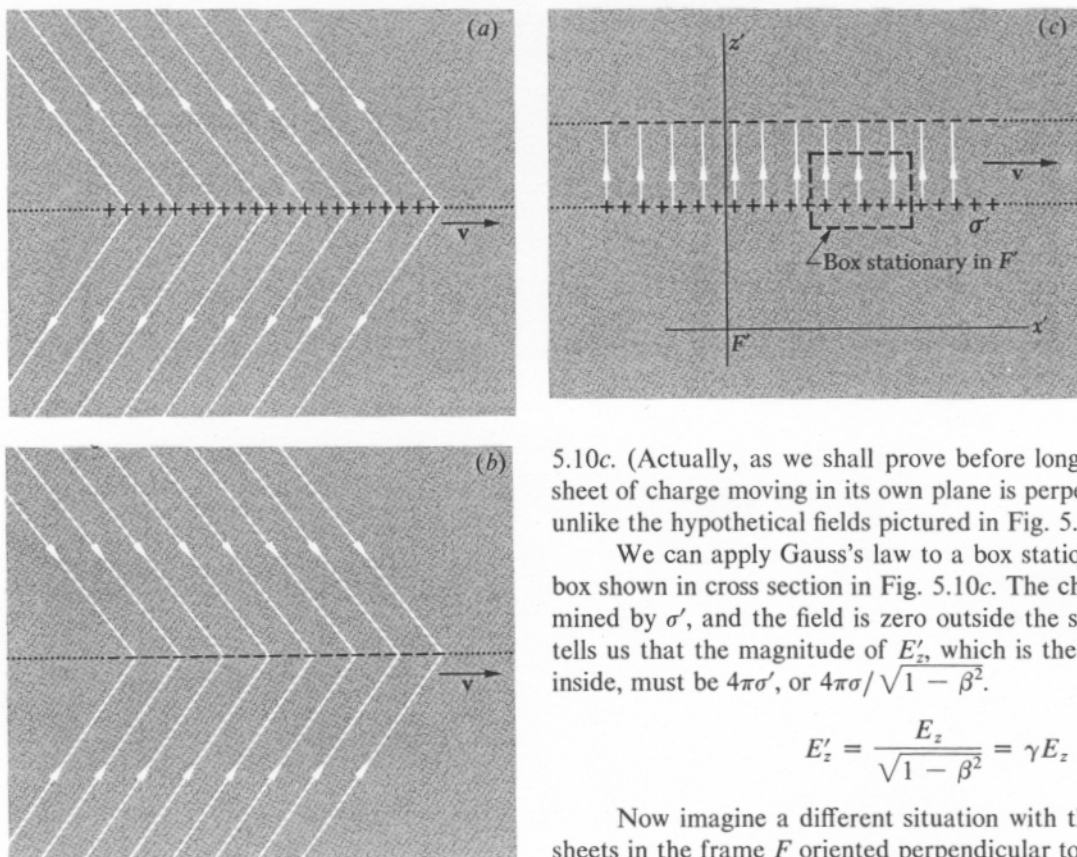
FIGURE 5.9

(a) Two square sheets of surface density  $+\sigma$  and  $-\sigma$ , stationary in an inertial frame  $F$ . (b) A cross-section view in the  $F$  frame.  $F'$  is a different frame moving in the  $-\hat{x}$  direction with respect to  $F$ . (c) Cross section of the charge sheets as seen in frame  $F'$ . Same charge is on shorter sheet, so charge density is greater:  $\sigma' = \gamma\sigma$ .

$E'$  means answering a question like this: If an observer in a certain inertial frame  $F$  measures an electric field  $E$  as so-and-so-many statvolts/cm, at a given point in space and time, what field will be measured at the same space-time point by an observer in a different inertial frame  $F'$ ? For a certain class of fields, we can answer this question by applying Gauss's law to some simple systems.

In the frame  $F$  (Fig. 5.9a) there are two stationary sheets of charge of uniform density  $\sigma$  and  $-\sigma$  esu/cm<sup>2</sup>, respectively. They are squares  $b$  cm on a side lying parallel to the  $xy$  plane, and their separation is supposed to be so small compared with their extent that the field between them can be treated as uniform. The magnitude of this field, as measured by an observer in  $F$ , is of course just  $4\pi\sigma$ . Now consider an inertial frame  $F'$  which moves toward the left, with respect to  $F$ , with velocity  $v$ . To an observer in  $F'$ , the charged "squares" are no longer square. Their  $x'$  dimension is contracted from  $b$  to  $b\sqrt{1-\beta^2}$ , where  $\beta$  stands for  $v/c$ , as usual. But total charge is invariant, that is, independent of reference frame, so the charge density measured in  $F'$  must be greater than  $\sigma$  in the ratio  $\gamma$ , that is,  $1/\sqrt{1-\beta^2}$ . Figure 5.9 shows the system in cross section, (b) as seen in  $F$  and (c) as seen in  $F'$ . What can we say about the electric field in  $F'$  if all we know about the electric field of moving charges is contained in Eq. 4?

For one thing, we can be sure that the electric field is zero outside the sandwich, and uniform between the sheets, at least in the limit as the extent of the sheets becomes infinite. The field of an infinite uniform sheet could not depend on the distance from the sheet, nor on position along the sheet. There is nothing in the system to fix a position along the sheet. But for all we know at this point, the field of a single moving sheet of positive charge might look like Fig. 5.10a. However, even if it did, the field of a sheet of negative charge moving with the same velocity would have to look like Fig. 5.10b, and the superposition of the two fields would still give zero field outside our two charged sheets and a uniform perpendicular field between them, as in Fig.



**FIGURE 5.10**

(a) Perhaps the field of a single moving sheet of positive charge looks like this. (It really doesn't, but we haven't proved that yet.) (b) If the field of the positive sheet looked like Fig. 5.10a, the field of a moving negative sheet would look like this. (c) The superposition of the fields of the positive and negative sheets would look like this, even if Fig. 5.10a and b were correct.

5.10c. (Actually, as we shall prove before long, the field of a single sheet of charge moving in its own plane is perpendicular to the sheet, unlike the hypothetical fields pictured in Fig. 5.10a and b.)

We can apply Gauss's law to a box stationary in frame  $F'$ , the box shown in cross section in Fig. 5.10c. The charge content is determined by  $\sigma'$ , and the field is zero outside the sandwich. Gauss's law tells us that the magnitude of  $E'_z$ , which is the only field component inside, must be  $4\pi\sigma'$ , or  $4\pi\sigma/\sqrt{1-\beta^2}$ .

$$E'_z = \frac{E_z}{\sqrt{1-\beta^2}} = \gamma E_z \quad (5)$$

Now imagine a different situation with the stationary charged sheets in the frame  $F$  oriented perpendicular to the  $x$  axis, as in Fig. 5.11. The observer in  $F$  now reports a field in the  $x$  direction of magnitude  $E_x = 4\pi\sigma$ . In this case, the surface charge density observed in the frame  $F'$  is the *same* as that observed in  $F$ . The sheets are not contracted; only the distance between them is contracted, but that doesn't enter into the determination of the field. This time we find by applying Gauss's law to the box stationary in  $F'$ :

$$E'_x = 4\pi\sigma' = 4\pi\sigma = E_x \quad (6)$$

That is all very well for the particularly simple arrangement of charges here pictured; do our conclusions have more general validity? This question takes us to the heart of the meaning of *field*. If the electric field  $\mathbf{E}$  at a point in space-time is to have a unique meaning, then the way  $\mathbf{E}$  appears in other frames of reference, in the same space-time neighborhood, cannot depend on the nature of the sources, wherever they may be, that produced  $\mathbf{E}$ . In other words, the observer in  $F$ , having measured the field in his neighborhood at some time, ought to be able to predict *from these measurements alone* what observers in other frames of reference would measure at the same space-time point. Were this not true, *field* would be a useless concept. The evi-

dence that it is true is the eventual agreement of our field theory with experiment.

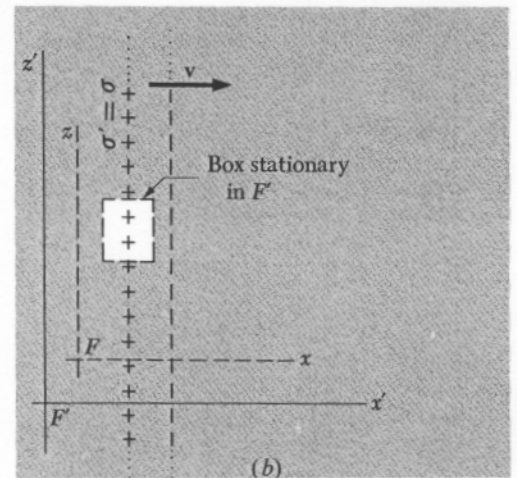
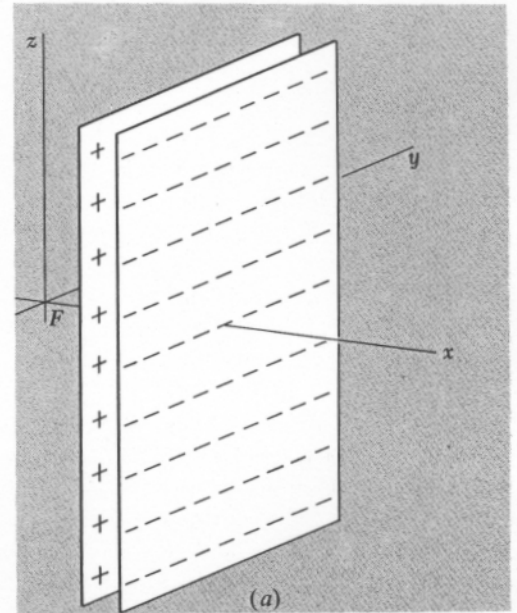
Seen in this light, the relations expressed in Eqs. 5 and 6 take on a significance beyond the special case of charges on parallel sheets. Consider any charge distribution, all parts of which are at rest with respect to the frame  $F$ . If an observer in  $F$  measures a field  $E_z$  in the  $z$  direction, then an observer in the frame  $F'$  will report, for the same space-time point, a field  $E'_z = \gamma E_z$ . That is, he will get a number, as the result of his  $E'_z$  measurement, which is larger by the factor  $\gamma$  than the number the  $F$  observer got in his  $E_z$  measurement. On the other hand, if the observer in  $F$  measures a field  $E_x$  in the  $x$  direction, the direction of the velocity of  $F'$  with respect to  $F$ , then the observer in  $F'$  reports a field  $E'_x$  equal to  $E_x$ . Obviously the  $y$  and the  $z$  directions are equivalent, both being transverse to the velocity  $v$ . Anything we have said about  $E'_z$  applies to  $E'_y$  too. Whatever the direction of  $\mathbf{E}$  in the frame  $F$ , we can treat it as a superposition of fields in the  $x$ , the  $y$ , and the  $z$  directions, and from the transformation of each of these predict the vector field  $\mathbf{E}'$  at that point in  $F'$ . Let's summarize this in words appropriate to relative motion in any direction: Charges at rest in frame  $F$  are the source of a field  $\mathbf{E}$ . Let frame  $F'$  move with speed  $v$  relative to  $F$ . At any point in  $F$ , resolve  $\mathbf{E}$  into a longitudinal component  $E_{\parallel}$  parallel to  $v$  and a transverse component  $E_{\perp}$  perpendicular to the direction of  $v$ . At the same space-time point in  $F'$ , the field  $\mathbf{E}'$  is to be resolved into  $E'_{\parallel}$  and  $E'_{\perp}$ ,  $E'_{\parallel}$  being parallel to  $v$  and  $E'_{\perp}$  perpendicular thereto. We have now learned that

$$\begin{aligned} E'_{\parallel} &= E_{\parallel} \\ E'_{\perp} &= \gamma E_{\perp} \end{aligned} \tag{7}$$

Our conclusion holds only for fields that arise from charges stationary in  $F$ . As we shall see presently, if charges in  $F$  are moving, the prediction of the electric field in  $F'$  involves knowledge of two fields in  $F$ , the electric and the magnetic. But we already have a useful result, one that suffices whenever we can find any inertial frame of reference in which all the charges remain at rest. We shall use it now to study the electric field of a point charge moving with constant velocity.

**FIELD OF A POINT CHARGE MOVING WITH CONSTANT VELOCITY**

**5.6** In the frame  $F$  the point charge  $Q$  remains at rest at the origin (Fig. 5.12a). At every point the electric field  $\mathbf{E}$  has the magnitude  $Q/r^2$  and is directed radially outward. In the  $xz$  plane its components at any point  $(x, z)$  are



**FIGURE 5.11** The electric field in another frame of reference (relative velocity parallel to field direction). (a) In reference frame  $F$ . (b) Cross-sectional view in reference frame  $F'$ .

$$E_x = \frac{Q}{r^2} \cos \theta = \frac{Qx}{(x^2 + z^2)^{3/2}} \quad (8)$$

$$E_z = \frac{Q}{r^2} \sin \theta = \frac{Qz}{(x^2 + z^2)^{3/2}}$$

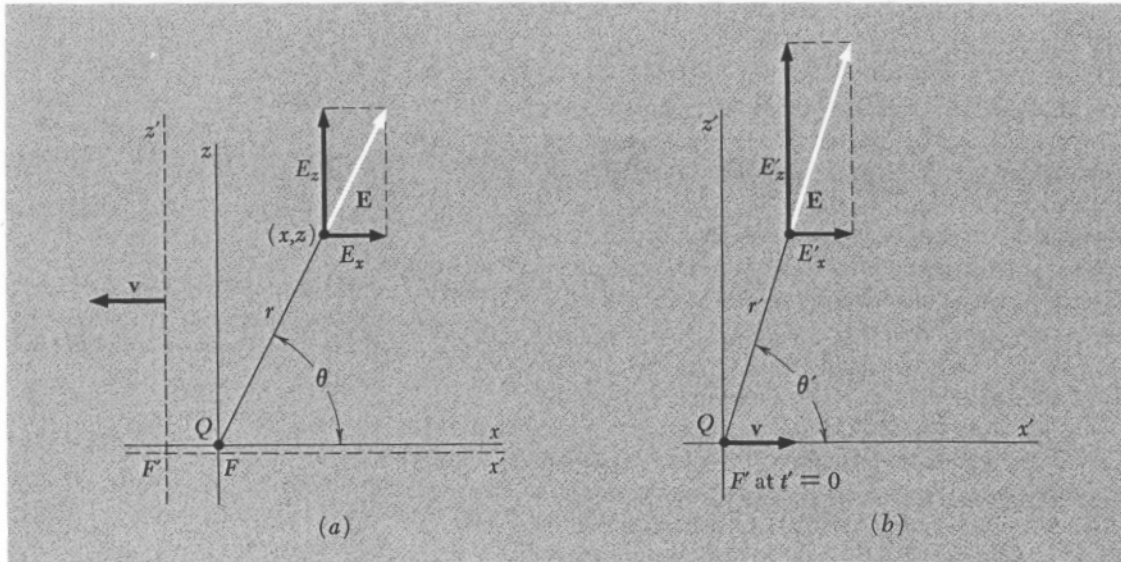
Consider another frame  $F'$  which is moving in the negative  $x$  direction, with speed  $v$ , with respect to frame  $F$ . We need the relation between the coordinates of an event in the two frames, for which we turn to the Lorentz transformation given in Eq. 2 of Appendix A. It simplifies the description to assume, as we are free to do, that the origins of the two frames coincide at time zero according to observers in both frames. In other words that event, the coincidence of the origins, can be the event  $A$  referred to by Eq. 2, with coordinates  $x_A = 0$ ,  $y_A = 0$ ,  $z_A = 0$ ,  $t_A = 0$  in frame  $F$  and  $x'_A = 0$ ,  $y'_A = 0$ ,  $z'_A = 0$ ,  $t'_A = 0$  in frame  $F'$ . Then event  $B$  is the space-time point we are trying to locate. We can omit the tag  $B$  and call its coordinates in  $F$  just  $x$ ,  $y$ ,  $z$ ,  $t$ , and its coordinates in  $F'$  just  $x'$ ,  $y'$ ,  $z'$ ,  $t'$ . Then Eq. 2 of Appendix A would become

$$x' = \gamma x - \gamma \beta ct \quad y' = y \quad z' = z \quad t' = \gamma t - \frac{\gamma \beta x}{c}$$

However, *that* transformation was for an  $F'$  frame moving in the positive  $x$  direction with respect to  $F$ , as one can quickly verify by noting that, with increasing time  $t$ ,  $x'$  gets smaller. To construct the Lorentz transformation for our problem, in which the  $F'$  frame moves in the opposite direction, we must either reverse the sign of  $\beta$  or switch the

**FIGURE 5.12**

The electric field of a point charge ( $a$ ) in a frame in which the charge is at rest, and ( $b$ ) in a frame in which the charge moves with constant velocity.



primes. We'll choose to do the latter because we want to express  $x$  and  $z$  in terms of  $x'$  and  $z'$ . The Lorentz transformation we need is therefore

$$x = \gamma x' - \gamma \beta c t' \quad y = y' \quad z = z' \quad t = \gamma t' - \frac{\gamma \beta x'}{c} \quad (9)$$

According to Eqs. 5 and 6,  $E'_z = \gamma E_z$  and  $E'_x = E_x$ . Using Eqs. 8 and 9, we can express the field components  $E'_z$  and  $E'_x$  in terms of the coordinates in  $F'$ . For the instant  $t' = 0$ , when  $Q$  passes the origin in  $F'$ , we have

$$E'_x = E_x = \frac{\gamma Q x'}{[(\gamma x')^2 + z'^2]^{3/2}} \quad (10)$$

$$E'_z = \gamma E_z = \frac{\gamma Q z'}{[(\gamma x')^2 + z'^2]^{3/2}}$$

Note first that  $E'_z/E'_x = z'/x'$ . This tells us that the vector  $\mathbf{E}'$  makes the same angle with the  $x'$  axis as does the radius vector  $\mathbf{r}'$ . Hence  $\mathbf{E}'$  points radially outward along a line drawn from the instantaneous position of  $Q$ , as in Fig. 5.12b. Pause a moment to let this conclusion sink in! It means that, if  $Q$  passed the origin of the primed system at precisely 12:00 noon, "prime time," an observer stationed anywhere in the primed system will report that the electric field in his vicinity was pointing, at 12:00 noon, exactly radially from the origin. This sounds at first like instantaneous transmission of information! How can an observer a mile away know where the particle is at the same instant? He can't. That wasn't implied. This particle, remember, has been moving at constant speed forever, on a "flight plan" that calls for it to pass the origin at noon. That information has been available for a long time. It is the *past history* of the particle that determined the field observed, if you want to talk about cause and effect. We'll inquire presently into what happens when there is an unscheduled change in the flight plan.

To find the strength of the field, we compute  $E_x'^2 + E_z'^2$ , which is the square of the magnitude of the field,  $E'^2$ .

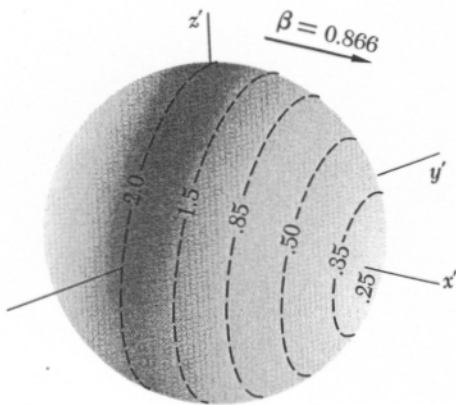
$$E'^2 = E_x'^2 + E_z'^2 = \frac{\gamma^2 Q^2 (x'^2 + z'^2)}{[(\gamma x')^2 + z'^2]^3} = \frac{Q^2 (x'^2 + z'^2)}{\gamma^4 [x'^2 + z'^2 - \beta^2 z'^2]^3}$$

$$= \frac{Q^2 (1 - \beta^2)^2}{(x'^2 + z'^2)^2 \left(1 - \frac{\beta^2 z'^2}{x'^2 + z'^2}\right)^3} \quad (11)$$

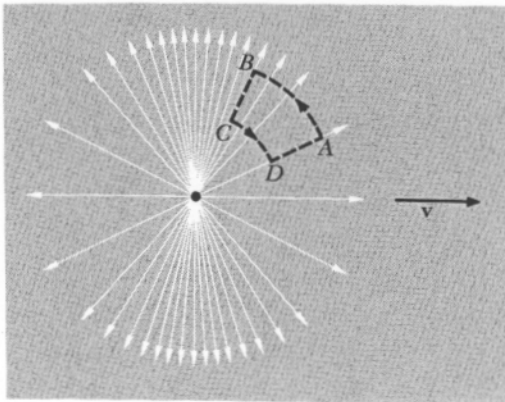
(Here, for once, it was neater with  $\beta$  worked back into the expression.) Let  $r'$  denote the distance from the charge  $Q$ , which is momentarily at the origin, to the point  $(x', z')$  where the field is measured:  $r' = (x'^2 + z'^2)^{1/2}$ . Let  $\theta'$  denote the angle between this radius vector and

**FIGURE 5.13**

The intensity in various directions of the field of a moving charge. At this instant, the charge is passing the origin of the  $x'y'z'$  frame. The numbers give the field strength relative to  $Q/r^2$ .

**FIGURE 5.14**

Another representation of the field of a uniformly moving charge.



the velocity of the charge  $Q$ , which is moving in the positive  $x'$  direction in the frame  $F'$ . Then since  $z' = r' \sin \theta'$ , the magnitude of the field can be written as

$$E' = \frac{Q}{r'^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta')^{3/2}} \quad (12)$$

There is nothing special about the origin of coordinates, nor about the  $x'z'$  plane as compared with any other plane through the  $x'$  axis. Therefore we can say quite generally that the electric field of a charge which has been in uniform motion is at a given instant of time directed radially from the instantaneous position of the charge, while its magnitude is given by Eq. 12 with  $\theta'$  the angle between the direction of motion of the charge and the radius vector from the instantaneous position of the charge to the point of observation.

For low speeds the field reduces simply to  $E' \approx Q/r'^2$ , and is practically the same, at any instant, as the field of a point charge stationary in  $F'$  at the instantaneous location of  $Q$ . But if  $\beta^2$  is not negligible, the field is stronger at right angles to the motion than in the direction of the motion, at the same distance from the charge. If we were to indicate the intensity of the field by the density of field lines, as is often done, the lines tend to concentrate in a pancake perpendicular to the direction of motion. Figure 5.13 shows the density of lines as they pass through a unit sphere, from a charge moving in the  $x'$  direction with a speed  $v/c = 0.866$ . A simpler representation of the field is shown in Fig. 5.14, a cross section through the field with some field lines in the  $x'z'$  plane indicated.†

This is a remarkable electric field. It is not spherically symmetrical, which is not surprising because in this frame there is a preferred direction, the direction of motion of the charge. However, the field is symmetrical about a plane perpendicular to the direction of motion of the charge. That, by the way, is sufficient to prove that the field of a uniform sheet of charge moving in its own plane must be perpendicular to the sheet. Think of that field as the sum of the fields of charge elements spread uniformly over the sheet. Since each of these individual fields has the fore-and-aft symmetry of Fig. 5.14 with respect to the direction of motion, their sum could only be perpendicular to the sheet. It could not look like Fig. 5.10a.

The field in Fig. 5.14 is a field that *no stationary charge distribution*, whatever its form, could produce. For in this field the line integral of  $E'$  is *not zero* around every closed path. Consider, for example,

†A two-dimensional diagram like Fig. 5.14 cannot faithfully represent the field intensity by the density of field lines. Unless we arbitrarily break off some of the lines, the density of lines in the picture will fall off as  $1/r'$ , whereas the intensity of the field we are trying to represent falls off as  $1/r'^2$ . So Fig. 5.14 gives only a qualitative indication of the variation of  $E'$  with  $r'$  and  $\theta'$ .

the closed path  $ABCD$  in Fig. 5.14. The circular arcs contribute nothing to the line integral, being perpendicular to the field; on the radial sections, the field is *stronger* along  $BC$  than along  $DA$ , so the *circulation* of  $\mathbf{E}'$  on this path is not zero. But remember, this is not an electrostatic field. In the course of time the electric field  $\mathbf{E}'$  at any point in the frame  $F'$  changes as the source charge moves.

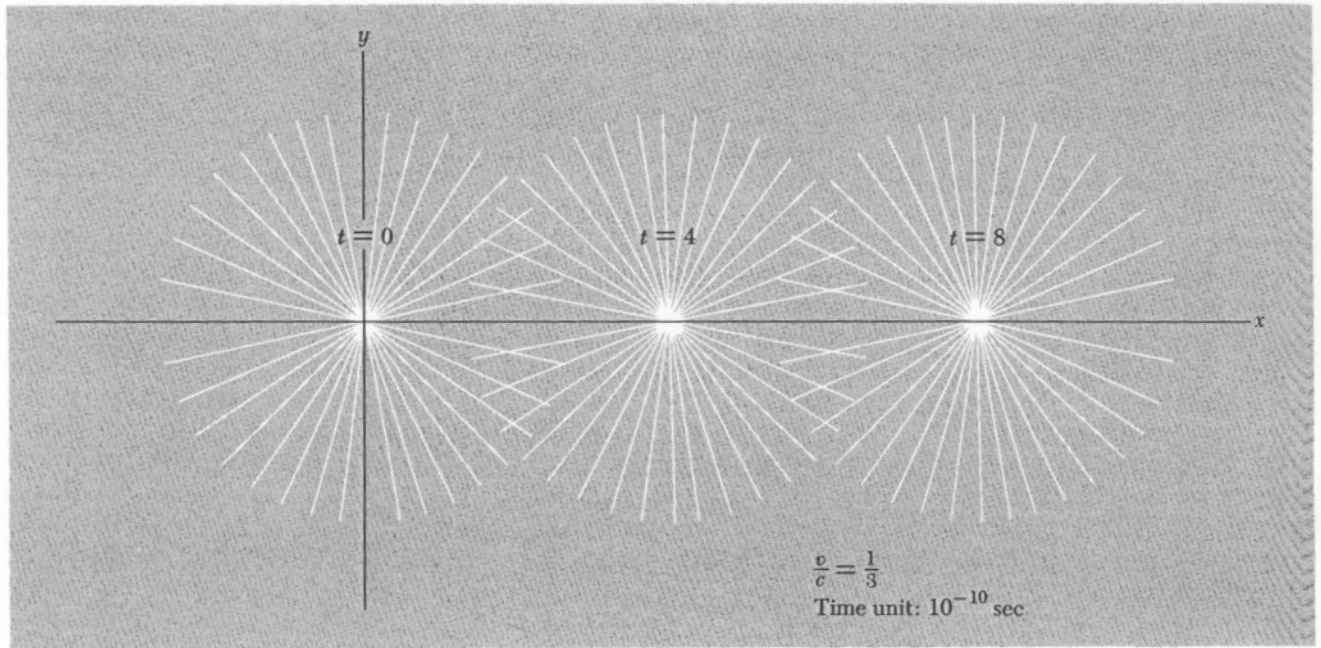
Figure 5.15 shows the electric field at certain instants of time observed in a frame of reference through which an electron is moving at constant velocity in the  $x$  direction.† In Figure 5.15, the speed of the electron is  $0.33c$ , its kinetic energy therefore about 30,000 eV [30 kiloelectron-volts (kev)]. The value of  $\beta^2$  is  $\frac{1}{9}$ , and the electric field does not differ greatly from that of a charge at rest. In Fig. 5.16, the speed is  $0.8c$ , corresponding to a kinetic energy of 335 kev. If the time unit for each diagram is taken as  $1.0 \times 10^{-10}$  sec, the distance scale is life-size, as drawn. Of course, the diagram holds equally well for *any* charged particle moving at the specified fraction of the speed of light. We mention the equivalent energies for an electron merely to remind the reader that relativistic speeds are nothing out of the ordinary in the laboratory.

#### FIELD OF A CHARGE THAT STARTS OR STOPS

**5.7** It must be clearly understood that *uniform velocity*, as we have been using the term, implies a motion at constant speed in a straight line that has been going on forever. What if our electron had *not* been traveling in the distant past along the negative  $x$  axis until it came into view in our diagram at  $t = 0$ ? Suppose it had been sitting quietly at rest at the origin, waiting for the clock to read  $t = 0$ . Just prior to  $t = 0$ , something gives the electron a sudden large acceleration, up to the speed  $v$ , and it moves away along the positive  $x$  axis at this speed. Its motion *from then on* precisely duplicates the motion of the electron for which Fig. 5.16 was drawn. But Fig. 5.16 does *not* correctly represent the field of the electron whose history was just described. To see that it cannot do so, consider the field at the point marked  $P$ , at time  $t = 2$ , which means  $2 \times 10^{-10}$  sec. In  $2 \times 10^{-10}$  sec a light signal travels 6 cm. Since this point lies more than 6 cm distant from the origin, it could not have received the news that the electron had started to move at  $t = 0$ ! Unless there is a gross violation of relativity—and we are taking the postulates of relativity as basis for this whole discussion—the field at the point  $P$  at time  $t = 2$ , and indeed at all points outside the sphere of radius 6 cm centered on the origin, *must be the field of a charge at rest at the origin.*

On the other hand, close to the moving charge itself, what hap-

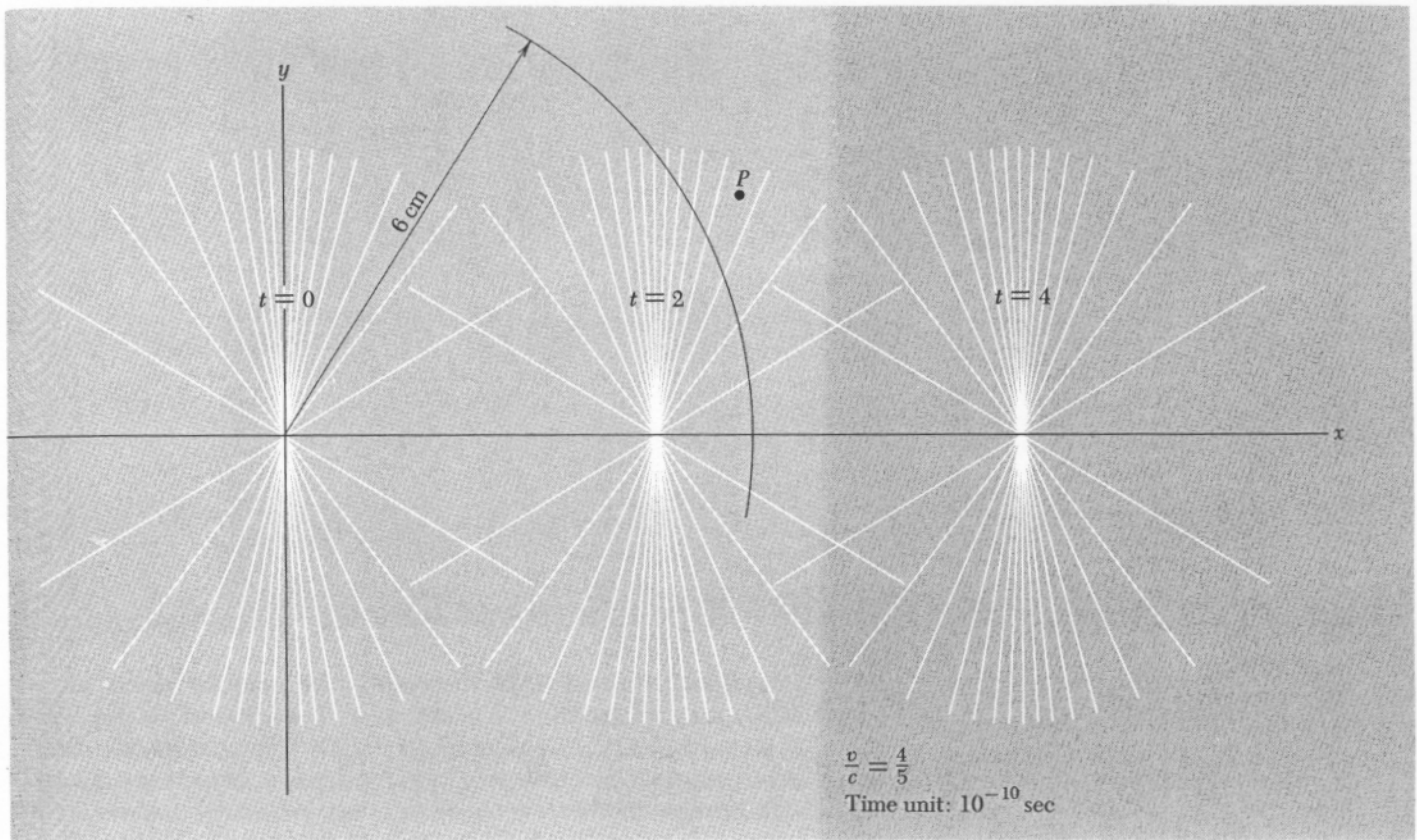
†Previously we had the charge at rest in the unprimed frame, moving in the primed frame. Here we adopt  $xyz$  for the frame in which the charge is moving, to avoid cluttering the subsequent discussion with primes.

**FIGURE 5.15**

The electric field of a moving charge, shown for three instants of time;  $v/c = 1/3$ .

pened in the remote past can't make any difference. The field must somehow change, as we consider regions farther and farther from the charge, at the given instant  $t = 2$ , from the field shown in the second diagram of Fig. 5.16 to the field of a charge at the origin. We can't deduce more than this without knowing how fast the news *does* travel. Suppose—just suppose—it travels as fast as it can without conflicting with the relativity postulates. Then if the period of acceleration is neglected, we should expect the field within the entire 6-cm-radius sphere, at  $t = 2$ , to be the field of a uniformly moving point charge. If that is so, the field of the electron which starts from rest, suddenly acquiring the speed  $v$  at  $t = 0$ , must look something like Fig. 5.17. There is a thin spherical shell (whose thickness in an actual case will depend on the duration of the interval required for acceleration) within which the transition from one type of field to the other takes place. This shell simply expands with speed  $c$ , its center remaining at  $x = 0$ . The arrowheads on the field lines indicate the direction of the field when the source is a negative charge, as we have been assuming.

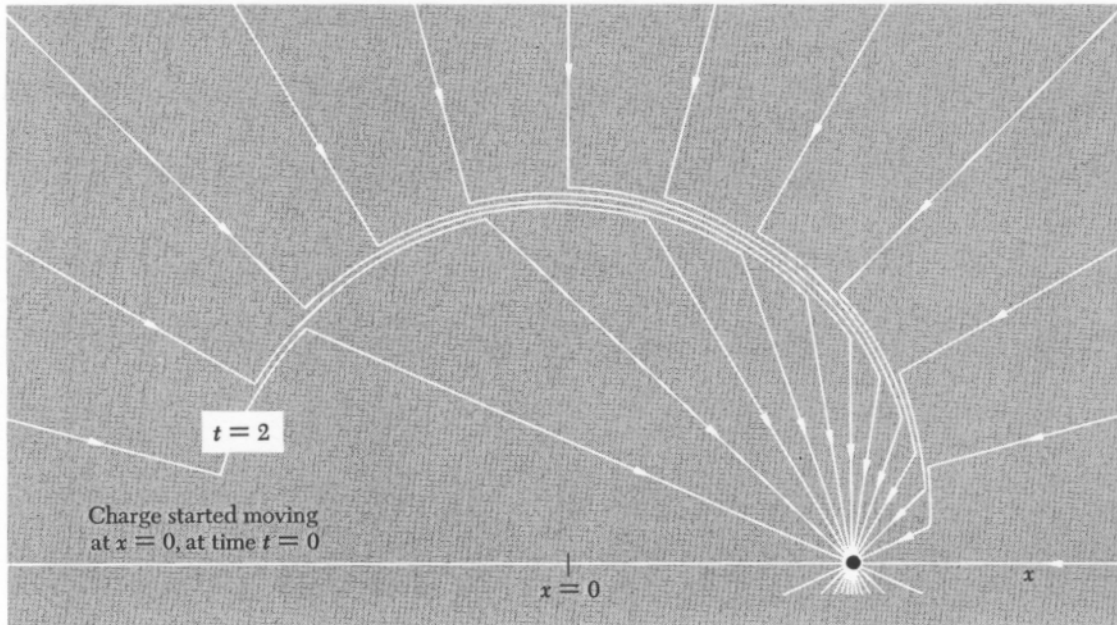
Figure 5.18 shows the field of an electron which had been moving with uniform velocity *until*  $t = 0$ , at which time it reached  $x = 0$  where it was abruptly *stopped*. Now the news that it was stopped cannot reach, by time  $t$ , any point farther than  $ct$  from the origin. The field outside the sphere of radius  $R = ct$  must be that which would have prevailed if the electron had kept on moving at its original speed. That is why we see the “brush” of field lines on the right in Fig. 5.18

**FIGURE 5.16**

The electric field of a moving charge, shown for three instants of time;  $v/c = 4/5$ .

pointing precisely down to the position where the electron would be if it hadn't stopped. (Note that this last conclusion does not depend on the assumption we introduced in the previous paragraph, that the news travels as fast as it can.) The field almost seems to have a life of its own!

It is a relatively simple matter to connect the inner and outer field lines. There is only one way it can be done that is consistent with Gauss's law. Taking Fig. 5.18 as an example, from some point such as  $A$  on the radial field line making angle  $\theta_0$  with the  $x$  axis, follow the field line wherever it may lead until you emerge in the outer field on some line making an angle that we may call  $\varphi_0$  with the  $x$  axis. (This line of course is radial from the extrapolated position of the charge, the apparent source of the outer field.) Connect  $A$  and  $D$  to the  $x$  axis by circular arcs, arc  $AE$  centered on the source of the inner field, arc  $DF$  centered on the apparent source of the outer field. Rotate the curve  $EABCDF$  around the  $x$  axis to generate a surface of revolution. As the surface encloses no charge, the surface integral of  $\mathbf{E}$  over the entire surface must be zero. The only contributions to the

**FIGURE 5.17**

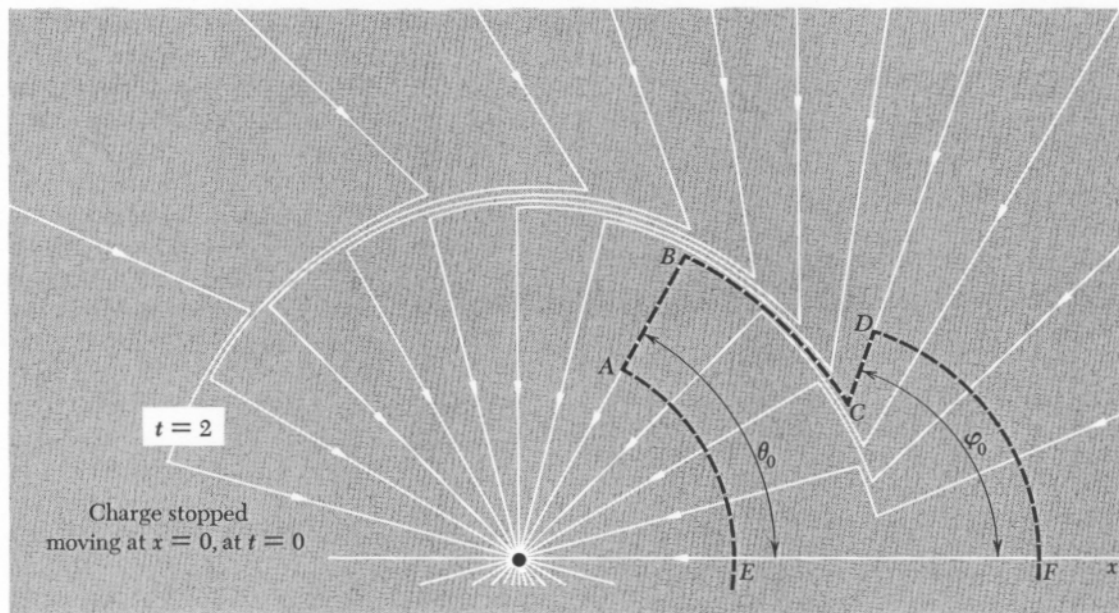
An electron initially at rest in the laboratory frame is suddenly accelerated at  $t = 0$  and moves with constant velocity thereafter. This is how the electric field looks at the instant  $t = 2$  all over the laboratory frame.

integral come from the spherical caps, for the surface generated by  $ABCD$  is parallel to the field by definition. The field over the inner cap is that of a point charge at rest at the origin. The field over the outer cap is the field, as given by Eq. 12, of a point charge moving with constant speed which would have been located, at this moment, at  $x = 2v$ . If you work Problem 5.11, you will find that the condition “flux in through one cap equals flux out through the other” requires

$$\tan \phi_0 = \gamma \tan \theta_0 \quad (13)$$

The presence of  $\gamma$  in this formula is not surprising. We had already noticed the “relativistic compression” of the field pattern of a rapidly moving charge, illustrated in Fig. 5.14. The important new feature in Fig. 5.18 is the zigzag in the field line  $ABCD$ . The cause of this is not the  $\gamma$  in Eq. 13, but the fact that the apparent source of the outer field is displaced from the source of the inner field. If  $AB$  and  $CD$  belong to the same field line, the connecting segment  $BC$  has to run *nearly perpendicular* to a radial vector. We have a *transverse* electric field there, and one that, to judge by the crowding of the field lines, is relatively intense compared with the radial field. As time goes on, the zigzag in the field lines will move radially outward with speed  $c$ . But the thickness of the shell of transverse field will not increase, for that was determined by the duration of the deceleration process.

The ever-expanding shell of transverse electric field would keep on going *even if* at some later time—at  $t = 3$ , say—we suddenly accelerated the electron back to its original velocity. That would only

**FIGURE 5.18**

An electron that has been moving with constant velocity reaches the origin at  $t = 0$ , is abruptly stopped, and remains at rest thereafter. This is how the field looks in the laboratory frame at the instant  $t = 2$ . The dashed outline follows a field line from  $A$  to  $D$ . Rotating the whole outline  $EABCD$  around the  $x$  axis generates a closed surface, the total flux through which must be zero. The flux in through the spherical cap  $FD$  must equal the flux out through the spherical cap  $EA$ . This condition suffices to determine the relation between  $\theta_0$  and  $\phi_0$ .

launch a new outgoing shell, this one looking very much like the field in Fig. 5.17. The field *does* have a life of its own! What has been created here before our eyes is an *electromagnetic wave*. The magnetic field that is also part of it was not revealed in this view. Later, in Chapter 9, we shall learn how the electric and magnetic fields work together in propagating an electrical disturbance through empty space. What we have discovered here is that such waves *must* exist if nature conforms to the postulates of special relativity and if electric charge is a relativistic invariant.

More can be done with our “zigzag-in-the-field-line” analysis. Appendix B shows how to derive, rather simply, an accurate and simple formula for the rate of radiation of energy by an accelerated electric charge. We must return now to the uniformly moving charge, which has more surprises in store.

### FORCE ON A MOVING CHARGE

**5.8** Equation 12 tells us the force experienced by a stationary charge in the field of another charge that is moving at constant velocity. We now ask a different question: What is the force that acts on a moving charge, one that moves in the field of some other charges?

We shall look first into the case of a charge moving through the field produced by stationary charges. It might be an electron moving between the charged plates of an oscilloscope, or an alpha particle moving through the Coulomb field around an atomic nucleus. The

sources of the field, in any case, are all at rest in some frame of reference which we shall call the "lab frame." At some place and time in the lab frame we observe a particle carrying charge  $q$  which is moving, at that instant, with velocity  $\mathbf{v}$  through the electrostatic field. What force appears to act on  $q$ ?

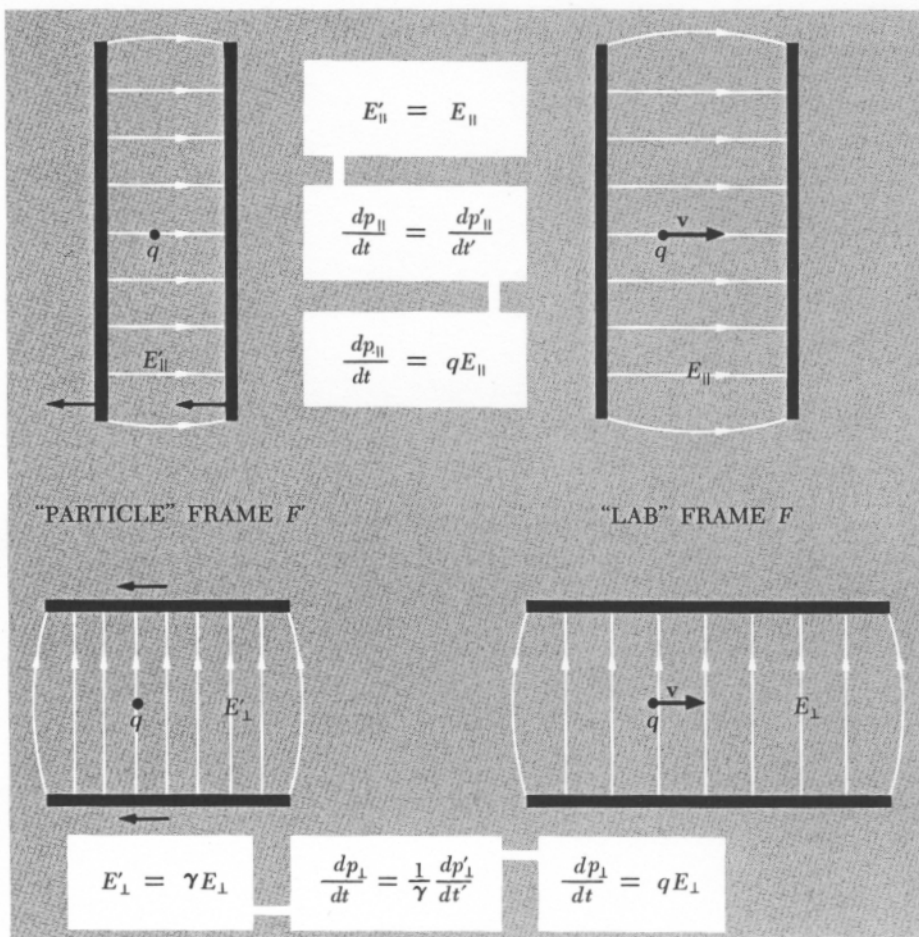
Force means rate of change of momentum, so we are really asking, What is the rate of change of momentum of the particle,  $d\mathbf{p}/dt$ , at this place and time, as measured in our lab frame of reference? (That is all we mean by the force on a moving particle.) The answer is contained, by implication, in what we have already learned. Let's look at the system from a coordinate frame  $F'$  moving, at the time in question, along with the particle. In this "particle frame" the particle will be, at least momentarily, at rest. It is the other charges that are now moving. This is a situation we know how to handle. The charge  $q$  has the same value; charge is invariant. The force on the stationary charge  $q$  is just  $\mathbf{E}'q$ , where  $\mathbf{E}'$  is the electric field observed in the frame  $F'$ . We have learned how to find  $\mathbf{E}'$  when  $\mathbf{E}$  is given; Eq. 7 provides our rule. Thus knowing  $\mathbf{E}$ , we can find the rate of change of momentum of the particle as observed in  $F'$ . All that remains is to transform this quantity back to  $F$ . So our problem hinges on the question, How does force, that is, rate of change of momentum, transform from one inertial frame to another?

The answer to that question is worked out later and is expressed in Eqs. 12 and 13 of Appendix A. The force component *parallel* to the relative motion of the two frames has the *same* value in the moving frame as it does in the rest frame of the particle. A force component *perpendicular* to the relative frame velocity is always *smaller*, by  $1/\gamma$ , than its value in the particle's rest frame. Let us summarize this in Eq. 14 using subscripts  $\parallel$  and  $\perp$  to label momentum components, respectively, parallel to and perpendicular to the relative velocity of  $F'$  and  $F$ , as we did in Eq. 7.

$$\begin{aligned}\frac{dp_{\parallel}}{dt} &= \frac{dp'_{\parallel}}{dt'} \\ \frac{dp_{\perp}}{dt} &= \frac{1}{\gamma} \frac{dp'_{\perp}}{dt'}\end{aligned}\tag{14}$$

Note that this is not a symmetrical relation between the primed and unprimed quantities. The rest frame of the particle, which we have chosen to call  $F'$  in this case, is special. In it the magnitude of the transverse force component is greater than in any other frame.

Equipped with the force transformation law, Eq. 14, and the transformation law for electric field components, Eq. 7, we return now to our charged particle moving through the field  $\mathbf{E}$ , and we discover an astonishingly simple fact. Consider first  $E_{\parallel}$ , the component of  $\mathbf{E}$  parallel to the instantaneous direction of motion of our charged par-



**FIGURE 5.19**  
 In a frame in which the charges producing the field  $\mathbf{E}$  are at rest, the force on a charge  $q$  moving with any velocity is simply  $q\mathbf{E}$ .

ticle. Transform to a frame  $F'$  moving, at that instant, with the particle. In that frame the longitudinal electric field is  $E'_{\parallel}$ , and according to Eq. 7,  $E'_{\parallel} = E_{\parallel}$ . So the force  $dp'_{\parallel}/dt'$  is

$$\frac{dp'_{\parallel}}{dt'} = qE'_{\parallel} = qE_{\parallel} \quad (15)$$

Back in frame  $F$ , observers are measuring the longitudinal force, that is, the rate of change of the longitudinal momentum component,  $dp_{\parallel}/dt$ . According to Eq. 14,  $dp_{\parallel}/dt = dp'_{\parallel}/dt'$ , so in frame  $F$  the longitudinal force component they find is

$$\frac{dp_{\parallel}}{dt} = \frac{dp'_{\parallel}}{dt'} = qE_{\parallel} \quad (16)$$

Of course the particle does not *remain* at rest in  $F'$  as time goes on. It will be accelerated by the field  $\mathbf{E}'$ , and  $\mathbf{v}'$ , the velocity of the particle

in the inertial frame  $F'$ , will gradually increase from zero. However, as we are concerned with the instantaneous acceleration, only infinitesimal values of  $v'$  are involved anyway, and the restriction on Eq. 14 is rigorously fulfilled. For  $E_{\perp}$ , the transverse field component in  $F$ , the transformation is  $E'_{\perp} = \gamma E_{\perp}$ , so that  $dp'_{\perp}/dt' = qE'_{\perp} = q\gamma E_{\perp}$ . But on transforming the force back to frame  $F$  we have  $dp_{\perp}/dt = (1/\gamma)(dp'_{\perp}/dt')$ , so the  $\gamma$  drops out after all:

$$\frac{dp_{\perp}}{dt} = \frac{1}{\gamma} (\gamma E_{\perp} q) = qE_{\perp} \quad (17)$$

The message of Eqs. 16 and 17 is simply this: The force on a charged particle in motion through  $F$  is  $q$  times the electric field  $\mathbf{E}$  in that frame, *strictly independent* of the velocity of the particle. Figure 5.19 is a reminder of this fact, and of the way we discovered it.

You have already used this result earlier in the course, where you were simply told that the contribution of the electric field to the force on a moving charge is  $q\mathbf{E}$ . Because this is familiar and so simple, you may think it is obvious and we have been wasting our time proving it. Now we could have taken it as an empirical fact. It has been verified over an enormous range, up to velocities so close to the speed of light, in the case of electrons, that the factor  $\gamma$  is  $10^4$ . From that point of view it is a most remarkable law. Our discussion in this chapter has shown that this fact is also a direct consequence of charge invariance.

## INTERACTION BETWEEN A MOVING CHARGE AND OTHER MOVING CHARGES

**5.9** We know that there can be a velocity-dependent force on a moving charge. That force is associated with a *magnetic field*, the sources of which are electric currents, that is, other charges in motion. Oersted's experiment showed that electric currents could influence magnets, but at that time the nature of a magnet was totally mysterious. Soon Ampère and others unraveled the interaction of electric currents with each other, as in the attraction observed between two parallel wires carrying current in the same direction. This led Ampère to the hypothesis that a magnetic substance contains permanently circulating electric currents. If so, Oersted's experiment could be understood as the interaction of the galvanic current in the wire with the permanent microscopic currents which gave the compass needle its special properties. Ampère gave a complete and elegant mathematical formulation of the interaction of steady currents, and of the equivalence of magnetized matter to systems of permanent currents. His brilliant conjecture about the actual nature of magnetism in iron had to wait a century, more or less, for its ultimate confirmation.

Whether the magnetic manifestations of electric currents arose from anything *more* than the simple transport of charge was not clear

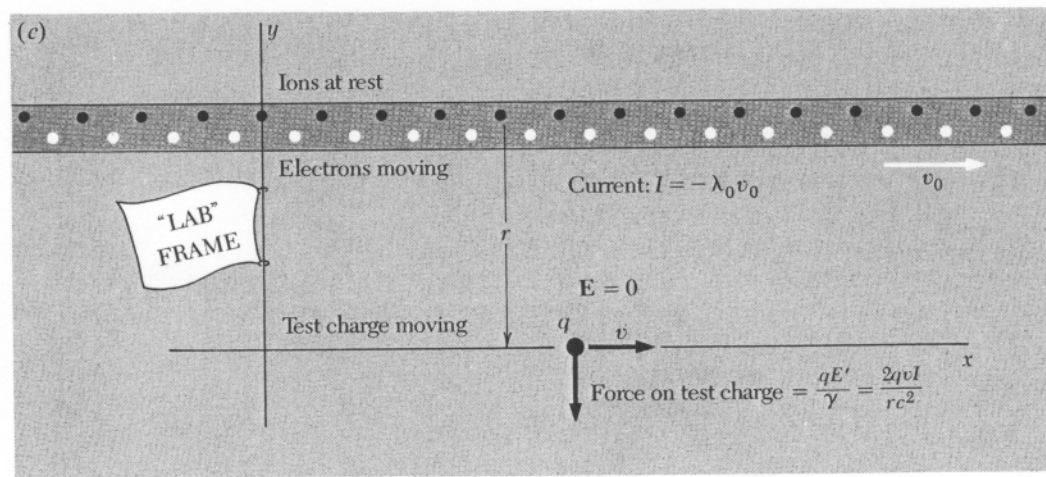
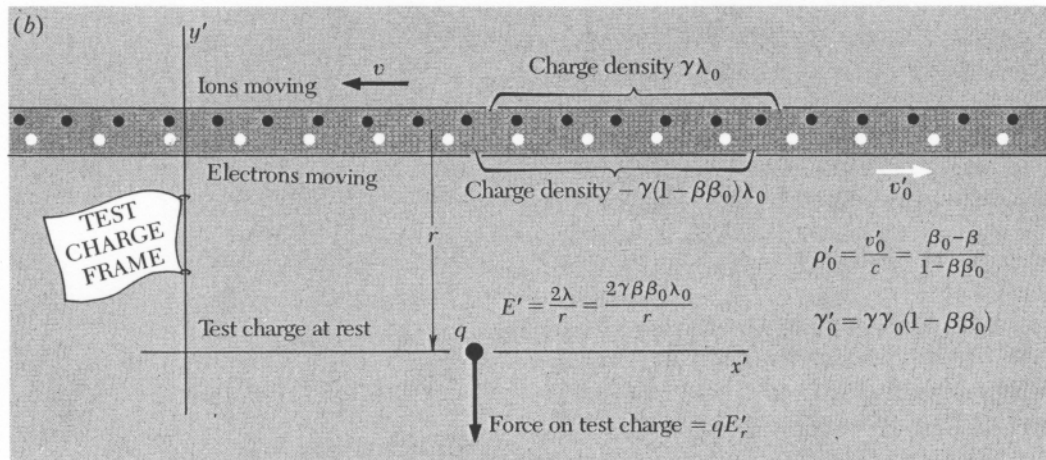
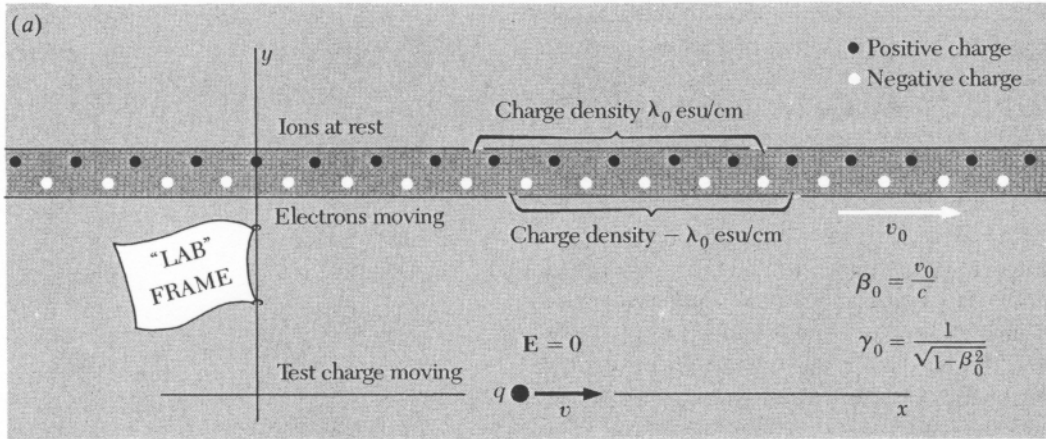
to Ampère and his contemporaries. Would the motion of an electrostatically charged object cause effects like those produced by a continuous galvanic current? Later in the century Maxwell's theoretical work suggested the answer should be *yes*. The first direct evidence was obtained by Henry Rowland, to whose experiment we shall return at the end of Chapter 6.

From our present vantage point, the magnetic interaction of electric currents can be recognized as an inevitable corollary to Coulomb's law. If the postulates of relativity are valid, if electric charge is invariant, and if Coulomb's law holds, then, as we shall now show, the effects we commonly call "magnetic" are bound to occur. They will emerge as soon as we examine the electric interaction between a moving charge and other moving charges. A very simple system will illustrate this.

In the lab frame of Fig. 5.20*a*, with spatial coordinates  $x, y, z$ , there is a line of positive charges, at rest and extending to infinity in both directions. We shall call them ions for short. Indeed, they might represent the copper ions that constitute the solid substance of a copper wire. There is also a line of negative charges that we shall call electrons. These are all moving to the right with speed  $v_0$ . In a real wire the electrons would be intermingled with the ions; we've separated them in the diagram for clarity. The linear density of positive charge is  $\lambda_0$  in esu/cm. It happens that the linear density of negative charge along the line of electrons is exactly equal in magnitude. That is, any given length of "wire" contains at a given instant the same number of electrons and protons.† The net charge on the wire is zero. Gauss' law tells us there can be no flux from a cylinder that contains no charge, so the electric field must be zero everywhere outside the wire. A test charge  $q$  at rest near this wire experiences no force whatever.

Suppose the test charge is not at rest in the lab frame but is moving with speed  $v$  in the  $x$  direction. Transform to a frame moving with the test charge, the  $x', y'$  frame in Fig. 5.20*b*. The test charge  $q$  is here at rest, but something else has changed: The wire appears to be charged! There are two reasons for that: The positive ions are closer together, and the electrons are farther apart. Because the lab frame in which the positive ions are at rest is moving with speed  $v$ , the distance between positive ions as seen in the test charge frame is contracted by  $\sqrt{1 - v^2/c^2}$ , or  $1/\gamma$ . The linear density of positive charge in this frame is correspondingly greater; it must be  $\gamma\lambda_0$ . The density of negative charge takes a little longer to calculate, for the electrons were already moving with speed  $v_0$  in the lab frame. Hence their linear density in the lab frame, which was  $-\lambda_0$ , had already been increased

†It doesn't have to, but that equality can always be established, if we choose, by adjusting the number of electrons per unit length. We assume that has been done.



by a Lorentz contraction. In the electrons' own rest frame the negative charge density must have been  $-\lambda_0/\gamma_0$ , where  $\gamma_0$  is the Lorentz factor that goes with  $v_0$ .

Now we need the speed of the electrons in the test charge frame in order to calculate their density there. To find that velocity ( $v'_0$  in Fig. 5.20*b*) we must add the velocity  $-v$  to the velocity  $v_0$ , remembering to use the relativistic formula for the addition of velocities (Eq. 6 in Appendix A). Let  $\beta'_0 = v'_0/c$ ,  $\beta_0 = v_0/c$ , and  $\beta = v/c$ . Then

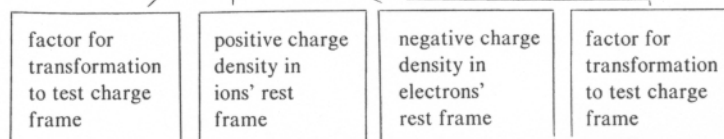
$$\beta'_0 = \frac{\beta_0 - \beta}{1 - \beta\beta_0} \quad (18)$$

The corresponding Lorentz factor  $\gamma'_0$ , obtained from Eq. 18 with a little algebra, is

$$\gamma'_0 = (1 - \beta_0'^2)^{-1/2} = \gamma\gamma_0(1 - \beta\beta_0) \quad (19)$$

This is the factor by which the linear density of negative charge in the electrons' own rest frame is enhanced when it is measured in the test charge frame. The total linear density of charge in the wire in the test charge frame,  $\lambda'$ , can now be calculated:

$$\lambda' = \gamma\lambda_0 - \frac{\lambda_0}{\gamma_0} \underbrace{\gamma\gamma_0(1 - \beta\beta_0)}_{\text{negative charge density in electrons' rest frame}} = \gamma\beta\beta_0\lambda_0 \quad (20)$$



The wire is positively charged. Gauss's law guarantees the existence of a radial electric field  $E'_r$  given by our familiar formula for the field of any infinite line charge:

$$E'_r = \frac{2\lambda'}{r'} = \frac{2\gamma\beta\beta_0\lambda_0}{r'} \quad (21)$$

At the location of the test charge  $q$  this field is in the  $-y'$  direction. The test charge will experience a force

$$F'_y = qE'_y = -\frac{2q\gamma\beta\beta_0\lambda_0}{r'} \quad (22)$$

Now let's return to the lab frame, pictured again in Fig. 5.20*c*. What is the magnitude of the force on the charge  $q$  as measured there? If its value is  $qE'_y$  in the rest frame of the test charge, observers in the lab frame will report a force smaller by the factor  $1/\gamma$ . Since  $r = r'$ , the force on our moving test charge, measured in the lab frame, is

**FIGURE 5.20**

A test charge  $q$  moving parallel to a current in a wire. (a) In the lab frame the wire, in which the positive charges are fixed, is at rest. The current consists of electrons moving to the right with speed  $v_0$ . The net charge on the wire is zero. There is no electric field outside the wire. (b) In a frame in which the test charge is at rest the positive ions are moving to the left with speed  $v$  and the electrons are moving to the right with speed  $v'_0$ . The linear density of a positive charge is greater than the linear density of negative charge. The wire appears positively charged, with an external field  $E'_r$  which causes a force  $qE'_r$  on the stationary test charge  $q$ . (c) That force transformed back to the lab frame has the magnitude  $qE'_r/\gamma$ , which is proportional to the product of the speed  $v$  of the test charge and the current in the wire,  $-\lambda_0 v_0$ .

$$F_y = \frac{F'_y}{\gamma} = -\frac{2q\beta\beta_0\lambda_0}{r} \quad (23)$$

Now  $-\lambda_0 v_0$  or  $-\lambda_0 \beta_0 c$  is just the total current  $I$  in the wire, in the lab frame, for it is the amount of charge flowing past a given point per second. We'll call current positive if it is equivalent to positive charge flowing in the positive  $x$  direction. Our current in this example is negative. Our result can be written this way:

$$F_y = \frac{2I}{rc^2} qv_x \quad (24)$$

We have found that in the lab frame the moving test charge experiences a force in the  $y$  direction which is proportional to the current in the wire, and to the velocity of the test charge in the  $x$  direction.

It is a remarkable fact that the force on the moving test charge does not depend separately on the velocity or density of the charge carriers but only on the product,  $\beta_0 \lambda_0$  in our example, that determines the charge transport. If we have a certain current  $I$ , say  $10^7$  esu/sec which is the same as 3.3 milliamps, it does not matter whether this current is composed of high-energy electrons moving with 99 percent of the speed of light, of electrons in a metal executing nearly random thermal motions with a slight drift in one direction, or of charged ions in solution with positive ions moving one way, negatives the other. Or it could be any combination of these, as Problem 5.18 will demonstrate. Furthermore, the force on the test charge is strictly proportional to the velocity of the test charge  $v$ . Our derivation was in no way restricted to small velocities, either for the charge carriers in the wire or for the moving charge  $q$ . Equation 24 is exact, with no restrictions.

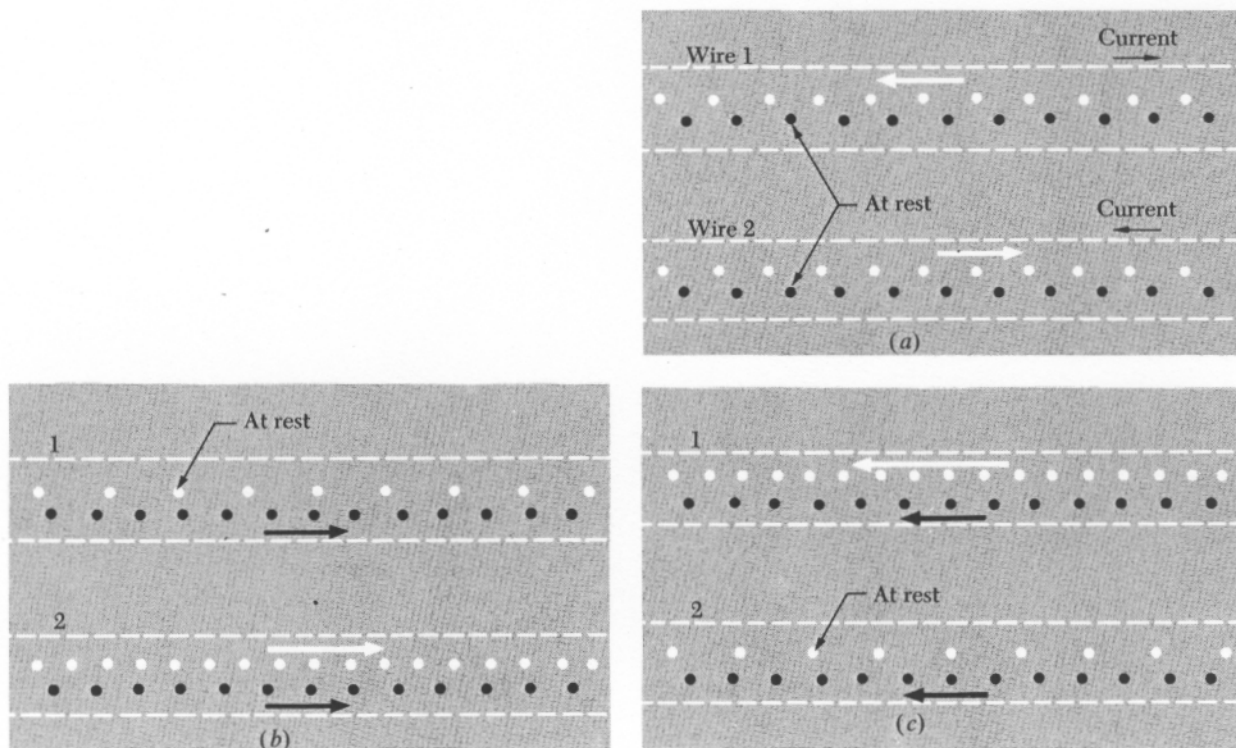
Let's see how this explains the mutual repulsion of conductors carrying currents in opposite directions, as shown in Fig. 5.1*b* at the beginning of this chapter. Two such wires are represented in the lab frame in Fig. 5.21*a*. Assume the wires are uncharged in the lab frame. Then there is no electrical force from the opposite wire on the positive ions which are stationary in the lab frame. Transferring to a frame in which one set of electrons is at rest (Fig. 5.21*b*), we find that in the other wire the electron distribution is Lorentz-contracted more than the positive ion distribution. Because of that the electrons at rest in this frame will be repelled by the other wire. But when we transfer to the frame in which those other electrons are at rest (Fig. 5.21*c*), we find the same situation. They too will be repelled. These repulsive forces will be observed in the lab frame as well, modified only by the factor  $\gamma$ . We conclude that the two streams of electrons will repel one another in the lab frame. The stationary positive ions, although they feel no direct electrical force from the other wire, will be the indirect bearers of this repulsive force if the electrons remain confined within

the wire. So the wires will be pushed apart, as in Fig. 5.1*b*, until some external force balances the repulsion.

Moving parallel to a current-carrying conductor, the charged particle experienced a force perpendicular to its direction of motion. What if it moves, instead, at right angles to the conductor? A velocity perpendicular to the wire will give rise to a force parallel to the wire—again, a force perpendicular to the particle's direction of motion. To see how this comes about, let us return to the lab frame of that system and give the test charge a velocity  $v$  in the  $y$  direction, as in Fig. 5.22*a*. Transferring to the rest frame of the test charge (Fig. 5.22*b*), we find the positive ions moving vertically downward. Certainly they cannot cause a horizontal field at the test-charge position. The  $x'$  component of the field from an ion on the left will be exactly cancelled by the  $x'$  component of the field of a symmetrically positioned ion on the right. The effect we are looking for is caused by the electrons. They are all moving obliquely in this frame, downward and toward the right. Consider the two symmetrically located electrons  $e_1$  and  $e_2$ . Their electric fields, relativistically compressed in the direction of the electrons' motion, are represented by a brush of field lines in the manner of Fig. 5.14. You can see that, although  $e_1$  and  $e_2$  are equally far away

**FIGURE 5.21**

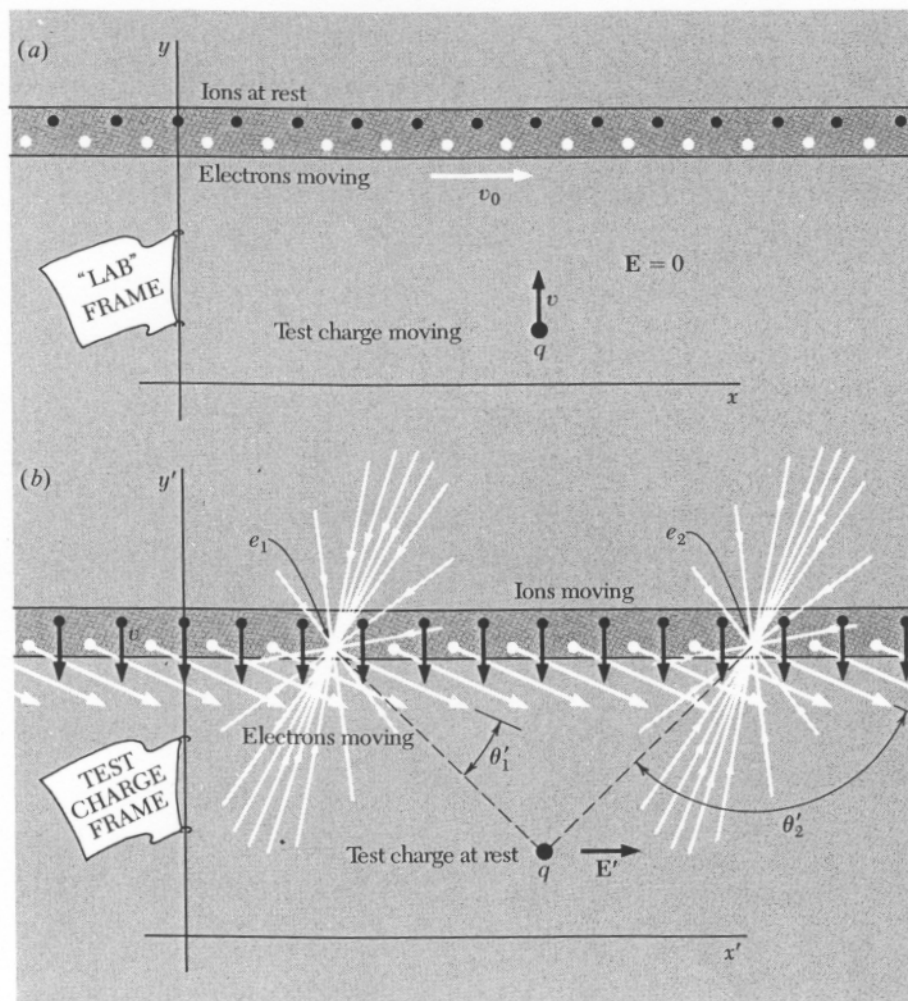
(*a*) Lab frame with two wires carrying current in opposite directions. As in metal wire, current is due to motion of negative ions (electrons) only. (*b*) Rest frame of electrons in wire 1. Note that in wire 2 positive ions are compressed, but electron distribution is contracted even more. (*c*) Rest frame of electrons in wire 2. Just as in (*b*), the other wire appears to rest to be negatively charged.



**FIGURE 5.22**

(a) The "wire" with its current of moving negative charges, or "electrons," is the same as in Fig. 5.20, but now the test charge is moving toward the wire. (b) In the rest frame of the test charge the positive charges, or "ions," are moving in the  $-\hat{y}$  direction. The electrons are moving obliquely. Because the field of a moving charge is stronger in directions more nearly perpendicular to its velocity, an electron on the right, such as  $e_2$ , causes a stronger field at the position of the test charge than does a symmetrically located electron on the left. Therefore the vector sum of the fields has in this frame a component in the  $\hat{x}'$  direction.

from the test charge, the field of electron  $e_2$  will be *stronger* than the field of electron  $e_1$  at that location. That is because the line from  $e_2$  to the test charge is more nearly perpendicular to the direction of motion of  $e_2$ . In other words, the angle  $\theta'$  that appears in the denominator of Eq. 12 is here different for  $e_1$  and  $e_2$ , so that  $\sin^2 \theta'_2 > \sin^2 \theta'_1$ . That will be true for any symmetrically located pair of electrons on the line, as you can verify with the aid of Fig. 5.23. The electron on the right always wins. Summing over all the electrons is therefore bound to yield a resultant field  $E'$  in the  $\hat{x}$  direction. The  $y'$  component of the electrons' field will be exactly cancelled by the field of the ions. That  $E'_y$  is zero is guaranteed by Gauss's law, for the number of charges per unit length of wire is the same as it was in the lab frame. The wire is uncharged in both frames.



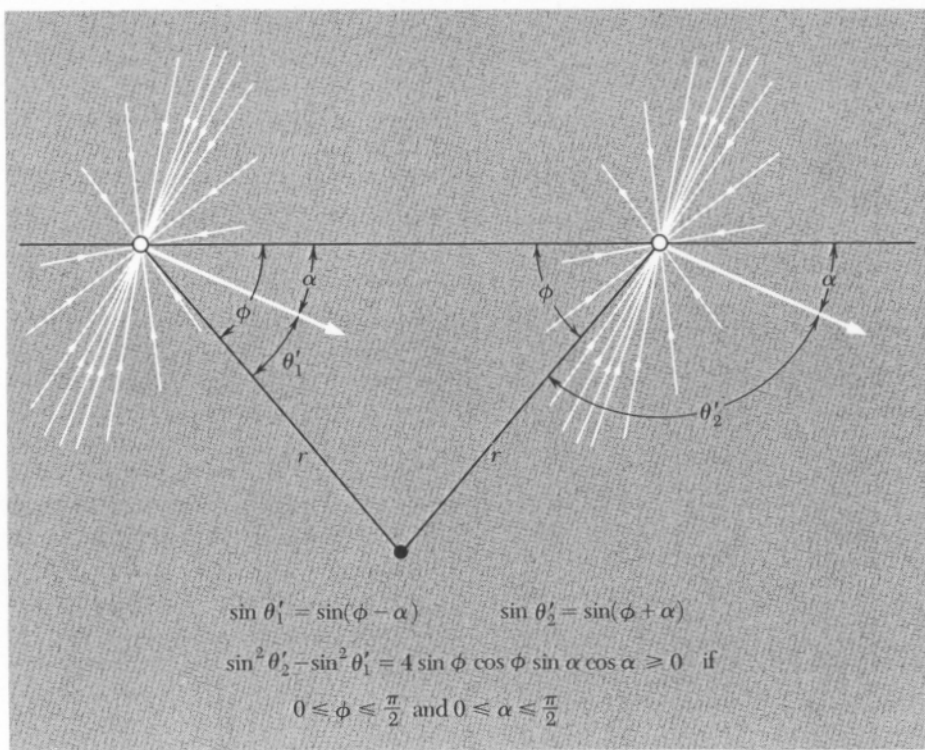
The force on our test charge,  $qE'_x$ , when transformed back into the lab frame, will be a force proportional to  $v$  in the  $\hat{x}$  direction, which is the direction of  $\mathbf{v} \times \mathbf{B}$  if  $\mathbf{B}$  is a vector in the  $\hat{z}$  direction, pointing at us out of the diagram. We could show that the magnitude of this velocity-dependent force is given here also by Eq. 24:  $F = 2qvI/rc^2$ . The physics needed is all in Eq. 12, but the integration is somewhat laborious and will not be undertaken here.

In this chapter we have seen how the fact of charge invariance implies forces between electric currents. That does not oblige us to look on one fact as the cause of the other. These are simply two aspects of electromagnetism whose relationship beautifully illustrates the more general law: Physics is the same in all inertial frames of reference.

If we had to analyze every system of moving charges by transforming back and forth among various coordinate systems, our task would grow both tedious and confusing. There is a better way. The overall effect of one current on another, or of a current on a moving charge, can be described completely and concisely by introducing a new field, the magnetic field.

**FIGURE 5.23**

A closer look at the geometry of Fig. 5.22b, showing that, for any pair of electrons equidistant from the test charge, the one on the right will have a larger value of  $\sin^2 \theta'$ . Hence, according to Eq. 5.12, it will produce the stronger field at the test charge.



## PROBLEMS

✓ **5.1** A capacitor consists of two parallel rectangular plates with a vertical separation of 2 cm. The east-west dimension of the plates is 20 cm, the north-south dimension is 10 cm. The capacitor has been charged by connecting it temporarily to a battery of 300 volts (1 stat-volt). How many excess electrons are on the negative plate? What is the electric field strength between the plates? Now give the following quantities as they would be measured in a frame of reference which is moving eastward, relative to the laboratory in which the plates are at rest, with speed  $0.6c$ : the three dimensions of the capacitor; the number of excess electrons on the negative plate; the electric field strength between the plates. Answer the same questions for a frame of reference which is moving upward with speed  $0.6c$ .

✓ **5.2** On a nylon filament 0.01 cm in diameter and 4 cm long there are  $5.0 \times 10^8$  extra electrons distributed uniformly over the surface. What is the electric field strength at the surface of the filament:

(a) In the rest frame of the filament?

(b) In a frame in which the filament is moving at a speed  $0.9c$  in a direction parallel to its length?

✓ **5.3** A beam of 9.5-megaelectron-volt (MeV) electrons ( $\gamma = 20$ ) amounting as current to 0.05 microamperes, is traveling through vacuum. The transverse dimensions of the beam are less than 1 mm, and there are no positive charges in or near it.

(a) In the lab frame, what is approximately the electric field strength 1 cm away from the beam, and what is the average distance between an electron and the next one ahead of it, measured parallel to the beam?

(b) Answer the same questions for the electron rest frame.

**5.4** Consider the trajectory of a charged particle which is moving with a speed  $0.8c$  in the  $x$  direction when it enters a large region in which there is a uniform electric field in the  $y$  direction. Show that the  $x$  velocity of the particle must actually *decrease*. What about the  $x$  component of momentum?

**5.5** Fixed in the frame  $F$  is a sheet of charge, of uniform surface density  $\sigma$ , which bisects the dihedral angle formed by the  $xy$  and the  $yz$  planes. The electric field of this stationary sheet is of course perpendicular to the sheet. How will this be described by observers in a frame  $F'$  that is moving in the  $x$  direction with velocity  $0.6c$  with respect to  $F$ ? What is the surface charge density  $\sigma'$  and what is the direction and strength of the electric field in  $F'$ ? Is it perpendicular to the sheet?

**5.6** In a colliding beam storage ring an antiproton going east passed a proton going west, the distance of closest approach being  $10^{-8}$  cm.

The kinetic energy of each particle in the lab frame was 93 Gev, corresponding to  $\gamma = 100$ . In the rest frame of the proton, what was the maximum intensity of the electric field at the proton due to the charge on the antiproton? For about how long, approximately, did the field exceed half its maximum intensity?

**5.7** The most extremely relativistic charged particles we know about are cosmic rays which arrive from outer space. Occasionally one of these particles has so much kinetic energy that it can initiate in the atmosphere a "giant shower" of secondary particles, dissipating, in total, as much as  $10^{19}$  ev of energy (more than 1 joule!). The primary particle, probably a proton, must have had  $\gamma \approx 10^{10}$ . How far away from such a proton would the field rise to 1 volt/meter as it passes? Roughly how thick is the "pancake" of field lines at that distance?

*Ans.* 4 meters;  $4 \times 10^{-10}$  meter.

**5.8** In the laboratory frame a proton is at rest at the origin at  $t = 0$ . At that instant a negative pion which has been traveling in along the  $x$  axis at a speed of  $0.6c$  has reached the point  $x = 0.01$  cm. There are no other charges around. What is the magnitude of the force on the pion? What is the magnitude of the force on the proton? What about Newton's third law?

**5.9** The deflection plates in a high-voltage cathode ray oscilloscope are two rectangular plates, 4 cm long and 1.5 cm wide, spaced 0.8 cm apart. There is a difference in potential of 6000 volts between the plates. An electron which has been accelerated through a potential difference of 250 kilovolts enters this deflector from the left, moving parallel to the plates and halfway between them, initially. We want to find the position of the electron and its direction of motion when it leaves the deflecting field at the other end of the plates. We shall neglect the fringing field and assume the electric field between the plates is uniform right up to the end. The rest mass of the electron may be taken as 500 kev. First carry out the analysis in the lab frame by answering these questions:  $\gamma = ?$ ;  $\beta = ?$ ;  $p_x$ , in units of  $mc$ , = ?; time spent between the plates = ? (Neglect the change in horizontal velocity discussed in Problem 5.4); transverse momentum component acquired, in units of  $mc$ , = ?; transverse velocity at exit = ?; vertical position at exit = ?; direction of flight at exit? Now describe this whole process as it would appear in an inertial frame which moved with the electron at the moment it entered the deflecting region: What do the plates look like? What is the field between them? What happens to the electron in this coordinate system? Your main object in this exercise is to convince yourself that the two descriptions are completely consistent.

**5.10** In the rest frame of a particle with charge  $q_1$  another particle with charge  $q_2$  is approaching, moving with velocity  $v$  not small compared with  $c$ . If it continues to move in a straight line, it will pass a

distance  $d$  from the position of the first particle. It is so massive that its displacement from the straight path during the encounter is small compared with  $d$ . Likewise, the first particle is so massive that its displacement from its initial position while the other particle is nearby is also small compared with  $d$ .

(a) Show that the increment in momentum acquired by each particle as a result of the encounter is perpendicular to  $\mathbf{v}$  and in magnitude  $2q_1q_2/vd$ . (Gauss's law can be useful here.)

(b) Expressed in terms of the other quantities, how large must the masses of the particles be to justify our assumptions?

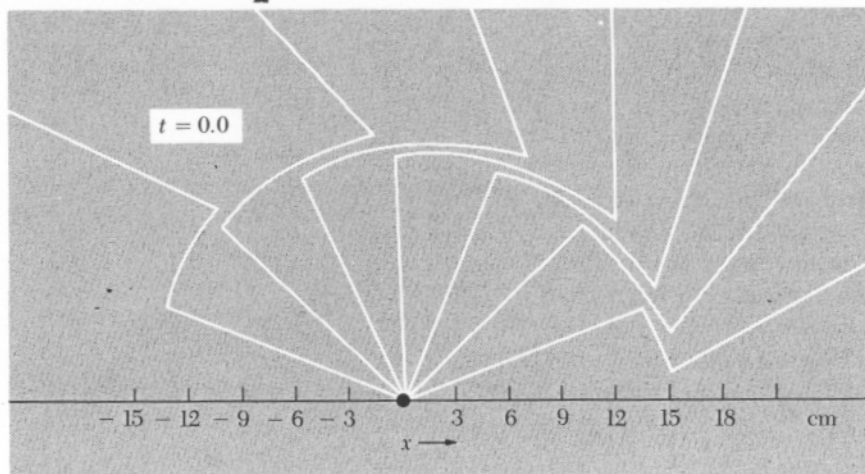
**5.11** Derive Eq. 13 by performing the integration to find the flux of  $E$  through each of the spherical caps described in the legend of Fig. 5.17. On the inner cap the field strength is constant, and the element of surface area may be taken as  $2\pi r^2 \sin \theta d\theta$ . On the outer cap the field is described by Eq. 12 with the appropriate changes in symbols, and the element of surface area is  $2\pi r^2 \sin \phi d\phi$ . The integral you will need is

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2(a^2 + x^2)^{1/2}}$$

**5.12** In the field of the moving charge  $Q$ , given by Eq. 12, we want to find an angle  $\delta$  such that half of the total flux from  $Q$  is contained between the two conical surfaces  $\theta' = \pi/2 + \delta$  and  $\theta' = \pi/2 - \delta$ . If you have done Problem 5.11 you have already done most of the work. You should find that, for  $\gamma \gg 1$ , the angle between the two cones is roughly  $1/\gamma$ .

**5.13** In the figure you see an electron at time  $t = 0.0$  and the associated electric field at that instant. Distances in centimeters are given.

**PROBLEM 5.13**



in the diagram.

(a) Describe what *has been* going on. Make your description as complete and quantitative as you can.

(b) Where was the electron at the time  $t = -7.5 \times 10^{-10}$  sec?

(c) What was the strength of the electric field at the origin at that instant?

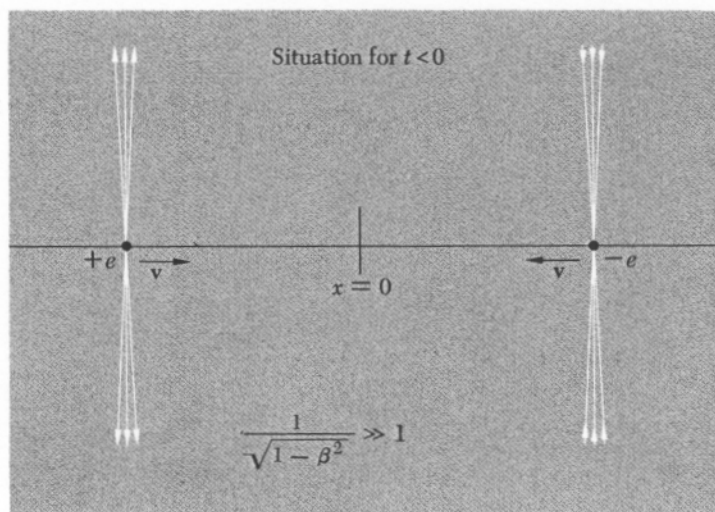
**5.14** The figure shows a highly relativistic positive particle approaching the origin from the left and a negative particle approaching with equal speed from the right. They collide at the origin at  $t = 0$ , find some way to dispose of their kinetic energy, and remain there as a neutral entity. What do you think the electric field looks like at some time  $t > 0$ ? Sketch the field lines. How does the field change as time goes on?

**5.15** In Fig. 5.20 the relative spacing of the black and white dots was designed to be consistent with  $\gamma = 1.2$  and  $\beta_0 = 0.8$ . Calculate  $\beta'_0$ . Find the value, as a fraction of  $\lambda_0$ , of the net charge density  $\lambda'$  in the test-charge frame.

**5.16** Suppose that the velocity of the test charge in Fig. 5.20 is made equal to that of the electrons,  $v_0$ . What would then be the linear densities of positive charge, and of negative charge, in the test-charge frame?

**5.17** Two protons are moving parallel to one another a distance  $r$  apart, with the same velocity  $\beta c$  in the lab frame. According to Eq. 12, at the instantaneous position of one of the protons the electric field  $E$  close to the other, as measured in the lab frame, is  $\gamma e/r^2$ . But the force on the proton measured in the lab frame is *not*  $\gamma e^2/r^2$ . Verify that by finding the force in the proton rest frame and transforming

**PROBLEM 5.14**



\_\_\_\_\_ sec between the plates and emerges, having acquired  $y$  momentum of magnitude  $p_y =$  \_\_\_\_\_ gm cm/sec. Its trajectory now slants upward at an angle  $\theta =$  \_\_\_\_\_ radians.

A fast neutron which just happened to be moving along with the electron when it passed through the anode reported subsequent events as follows: "We were sitting there when this capacitor came flying at us at \_\_\_\_\_ cm/sec. It was \_\_\_\_\_ cm long, so it surrounded us for \_\_\_\_\_ sec. That didn't bother me, but the electric field of \_\_\_\_\_ statvolts/cm accelerated the electron so that after the capacitor left us the electron was moving away from me at \_\_\_\_\_ cm/sec."