B

RADIATION BY AN ACCELERATED CHARGE

A particle with charge q has been moving in a straight line at constant speed v_0 for a long time. It runs into something, let us imagine, and in a short period of constant deceleration, of duration τ, the particle is brought to rest. The graph of velocity versus time in Fig. B.1 describes its motion. What must the electric field of this particle look like after that? Figure B.2 shows how to derive it.

We shall assume that v_0 is small compared with c. Let $t = 0$ be the instant the deceleration began, and let $x = 0$ be the position of the particle at that instant. By the time the particle has completely stopped it will have moved a little farther on, to $x = \frac{1}{2}v_0\tau$. That distance, although we tried to indicate it on our diagram, is small compared with the other distances that will be involved.

We now examine the electric field at a time $t = T \gg \tau$. Observers farther away from the origin than $R = cT$ cannot have learned that the particle was decelerated. Throughout that region, region I in Fig. B.2, the field must be that of a charge which has been moving *and is still moving* at the constant speed v_0. That field, as we discovered in Section 5.7, appears to emanate from the present position of the charge, which for an observer anywhere in region I is the point $x = v_0T$ on the x axis. That is where the particle would be now if it hadn't been decelerated. On the other hand, for any observer whose distance from the origin is less than $c(T - \tau)$, that is to say, for any observer in region II, the field is that of a charge at rest close to the origin (actually at $x = \frac{1}{2}v_0\tau$).

What must the field be like in the transition region, the spherical shell of thickness $c\tau$? Gauss's law provides the key. A field line such as AB lies on a cone around the x axis which includes a certain

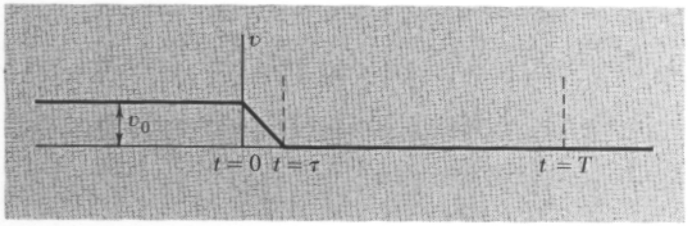

FIGURE B.1
Velocity-time diagram for a particle which traveled at constant speed v_0 until $t = 0$. It then experienced a constant negative acceleration of magnitude $a = v_0/\tau$, which brought it to rest at time $t = \tau$. We assume v_0 is small compared to c.

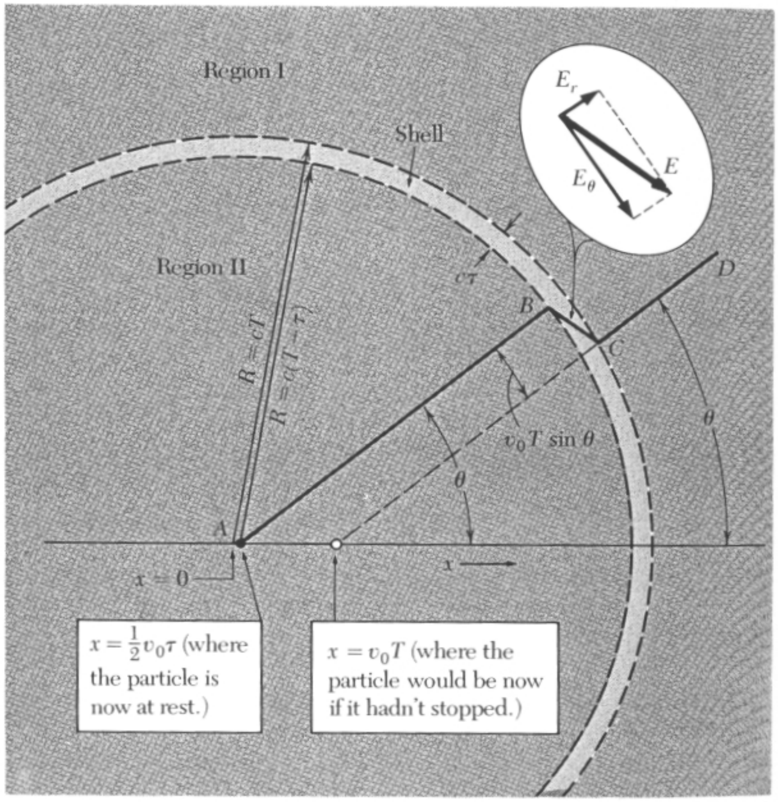

FIGURE B.2
Space diagram for the instant $t = T \gg \tau$, a long time after the particle has stopped. For observers in region *I*, the field must be the position $x = v_0T$; for observers in region II, it is that of a particle at rest close to the origin. The transition region is a shell of thickness $c\tau$.

amount of flux from the charge q. If CD makes the same angle θ with the axis, the cone on which it lies includes that same amount of flux. (Because v_0 is small, the relativistic compression of field lines visible in Fig. 5.13 and 5.17 is here negligible.) Hence AB and CD must be parts of the same field line, connected by a segment BC. This tells us the *direction* of the field **E** within the shell; it is the direction of the line segment BC. This field **E** within the shell has both a radial component E_r and a transverse component E_θ. From the geometry of the figure their ratio is easily found.

$$\frac{E_\theta}{E_r} = \frac{v_0 T \sin\theta}{c\tau} \qquad (1)$$

Now E_r must have the same value within the shell thickness that it does in region II near B. (Gauss's law again!) Therefore $E_r = q/R^2 = q/c^2T^2$, and substituting this in Eq. 1 we obtain

$$E_\theta = \frac{v_0 T \sin\theta}{c\tau} \qquad E_r = \frac{qv_0 \sin\theta}{c^3 T\tau} \tag{2}$$

But $v_0/\tau = a$, the magnitude of the (negative) acceleration, and $cT = R$, so our result can be written

$$E_\theta = \frac{qa \sin\theta}{c^2 R} \tag{3}$$

A remarkable fact is here revealed: E_θ is proportional to $1/R$, *not* to $1/R^2$! As time goes on and R increases, the transverse field E_θ will eventually become very much stronger than E_r. Accompanying this transverse (that is, perpendicular to **R**) electric field will be a magnetic field of equal strength perpendicular to both **R** and **E**. This is a general property of an electromagnetic wave, explained in Chapter 9.

Let us calculate the energy stored in the transverse electric field above, in the whole spherical shell. The energy density is

$$\frac{E_\theta^2}{8\pi} = \frac{q^2a^2\sin^2\theta}{8\pi R^2 c^4} \tag{4}$$

The volume of the shell is $4\pi R^2 c\tau$, and the average value of $\sin^2\theta$ over a sphere† is ⅔. The total energy of the transverse electric field is therefore

$$\frac{2}{3}\,4\pi R^2 c\tau\,\frac{q^2a^2}{8\pi R^2 c^4} = \frac{1}{3}\frac{q^2a^2\tau}{c^3}$$

To this we must add an equal amount for the energy stored in the transverse magnetic field:

$$\text{Total energy in transverse electromagnetic field} = \frac{2}{3}\frac{q^2a^2\tau}{c^3} \tag{5}$$

The radius R has canceled out. This amount of energy simply travels outward, undiminished, with speed c from the site of the deceleration. Since τ is the duration of the deceleration, and is also the duration of the electromagnetic pulse a distant observer measures, we can say that the *power* radiated during the acceleration process was

$$P_{\text{rad}} = \frac{2}{3}\frac{q^2a^2}{c^3} \tag{6}$$

†Our polar axis in this figure is the x axis: $\cos^2\theta = x^2/R^2$. With a bar denoting an average over the sphere, $\overline{x^2} = \overline{y^2} = \overline{z^2} = \frac{1}{3}R^2$. Hence $\overline{\cos^2\theta} = \frac{1}{3}$, and $\overline{\sin^2\theta} = 1 - \overline{\cos^2\theta} = \frac{2}{3}$.

As it is the square of the instantaneous acceleration that appears in Eq. 6, it doesn't matter whether a is positive or negative. Of course it ought not to, for stopping in one inertial frame could be starting in another. Speaking of different frames, P_{rad} itself turns out to be Lorentz-invariant, which is sometimes very handy. That is because P_{rad} is *energy/time*, and energy transforms like time, each being the fourth component of a four-vector, as noted in Appendix A.

We have here a more general result than we might have expected. Equation 6 correctly gives the instantaneous rate of radiation of energy by a charged particle moving with variable acceleration—for instance, a particle vibrating in simple harmonic motion. It applies to a wide variety of radiating systems from radio antennas to atoms and nuclei.

PROBLEMS

B.1 An electron moving initially at constant speed v is brought to rest with uniform deceleration a lasting for a time $t = v/a$. Compare the electromagnetic energy radiated during the deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time t and the classical electron radius r_0, defined as e^2/mc^2.

B.2 An elastically bound electron vibrates in simple harmonic motion at frequency ω with amplitude A.

(*a*) Find the average rate of loss of energy by radiation.

(*b*) If no energy is supplied to make up the loss, how long will it take for the oscillator's energy to fall to $1/e$ of its initial value?

Ans. (*b*): $3mc^3/2e^2\omega^2$.

B.3 A plane electromagnetic wave with frequency ω and electric field amplitude E_0 is incident on an isolated electron. In the resulting sinusoidal oscillation of the electron the maximum acceleration is E_0e/m. How much power is radiated by this oscillating charge, averaged over many cycles? (Note that it is independent of the frequency ω.) Divide this average radiated power by $E_0^2c/8\pi$, the average power density (power per unit area of wavefront) in the incident wave. This gives a constant σ with the dimensions of area, called a *scattering cross section.* The energy radiated, or scattered, by the electron, and thus lost from the plane wave, is equivalent to that falling on an area σ. (The case here considered, involving a free electron moving nonrelativistically is often called *Thomson scattering* after J. J. Thomson, the discoverer of the electron, who first calculated it.)

B.4 Our master formula, Eq. 6, is useful for relativistically moving particles, even though we assumed $v_0 \ll c$ in the derivation. All we have to do is transform to an inertial frame F' in which the particle in question is, at least temporarily, moving slowly, apply Eq. 6 in that frame, then transform back to any frame we choose. Consider a highly relativistic electron ($\gamma \gg 1$) moving perpendicular to a magnetic field **B.** It is continually accelerated perpendicular to the field, and must radiate. At what rate does it lose energy? To answer this, transform to a frame F' moving momentarily along with the electron, find E' in that frame, and P'_{rad}. Now show that, because power is *energy/time*, $P_{\text{rad}} = P'_{\text{rad}}$. This radiation is generally called *synchrotron radiation.*

Ans. $P_{\text{rad}} = \frac{2}{3}\gamma^2 B^2 e^2 / m^2 c^3$.