# Coaxial Transmission Lines

**Honors Physics – P222**

*Alex R. Dzierba*

## Introduction

Understanding transmission lines brings together a lot of nice physics and an opportunity to make quantitative measurements. One of the more common transmission lines is the coaxial cable – used for cable TV, for example, and ubiquitous in high energy and nuclear physics experiments. A typical experiment literally uses miles of transmission lines. You used coaxial cables in the experiments in this class. The transmission line consists of an inner conductor – a wire – and an outer conductor – usually a metal braided jacket .

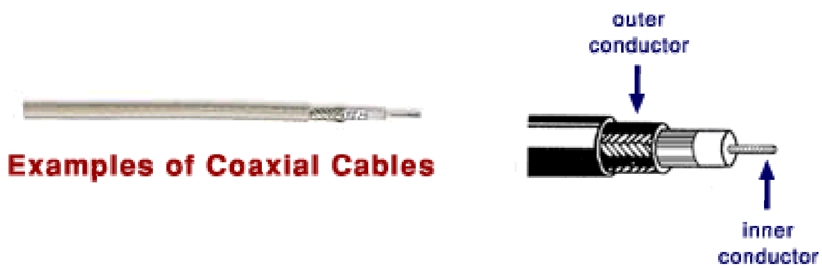


Figure 1: Anatomy of coaxial cables

The outer conductor is electrically connected to the connectors that you usually find at the end of the cable. Many of the cables we use have BNC connectors as can be seen in this photo of a RG58 cable with a characteristic impedance of 50 Ω and a propagation velocity of 0.7 *c*:

Figure 2: RG58 cable Source: http://www.oscaroscar.com/images/pp_RG58-10.jpg

We will picture the coaxial transmission line as:

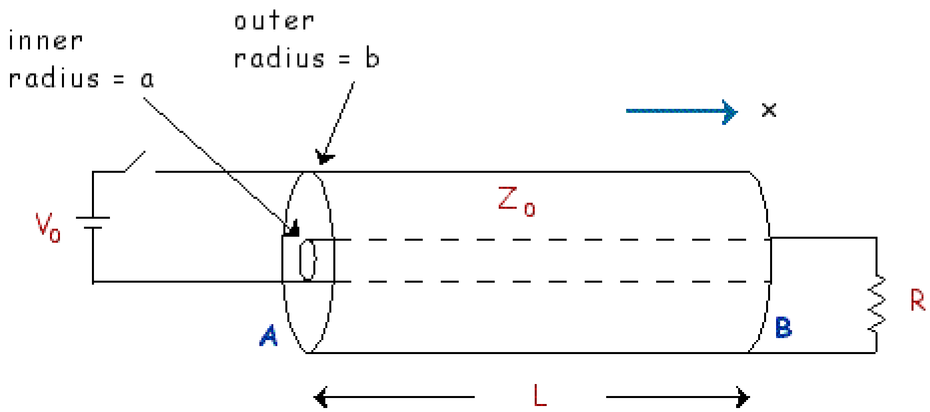


Figure 3: Coaxial cable schematic

A coaxial transmission line has a characteristic impedance, $Z_o$, and a characteristic propagation velocity. Both of these depend on the material in the space between the conductors. A coaxial cable has a certain capacitance per length, $C_o$, and an inductance per length, $L_o$ and the propagation velocity and impedance, we will show, are given by:

$$v = \frac{1}{\sqrt{L_o C_o}} \qquad (1)$$

$$Z_o = \sqrt{\frac{L_o}{C_o}} \qquad (2)$$

The calculation of the capacitance of per length and inductance per length for a coaxial cable is left as a problem. The result:

$$C_o = \frac{2\pi\varepsilon_o}{\ln(b/a)} \qquad (3)$$

$$L_o = \frac{\mu_o \ln(b/a)}{2\pi} \qquad (4)$$

This assumes that the space between the conductors is a vacuum. If a dielectric fills this space then $\varepsilon_o \to \kappa\varepsilon_o$ where $\kappa$ is the dielectric constant. Using these results:

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c \qquad (5)$$

$$Z_o = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon_o}} \qquad (6)$$

We note that in our SI units:

$$\sqrt{\frac{\mu_o}{\varepsilon_o}} \approx 377 \ \ \Omega$$

In the drawing of Fig. 3 we show the length of the cable as L but often cable lengths are given in units of nanoseconds that actually measures the delay time through the cable. For completeness, the equivalent diagram of the above is shown here:

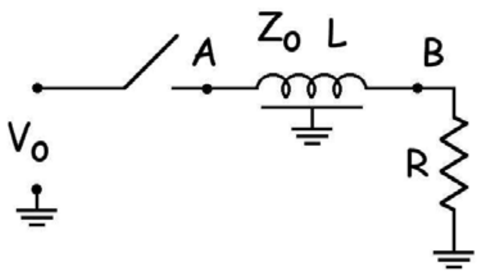


Figure 4: Circuit diagram with a coaxial cable.

**Transmission of Voltage and Current in a Cable**

When the switch in Fig 3 is closed, the voltage and current propagates down the cable with a finite velocity which is less than or equal to the speed of light, *c*.

Before we continue it is interesting to look at what is going on from the point of voltage and current (V and I) or the electric and magnetic fields (**E** and **B**). Applying a voltage between the two conductors at one end leads to a current and an **E** field. The current at the end also leads to a **B** field. The fields do not turn on instantaneously everywhere inside the cable but propagate down the cable (as does the voltage and current). The directions of **E** and **B** are also such that they are perpendicular to each other and so that the Poynting vector, $\varepsilon_o c^2(\mathbf{E} \times \mathbf{B})$, points in the propagation direction. Behind the wave front is stored energy in the **E** and **B** fields filling the volume between the space between the conductors.

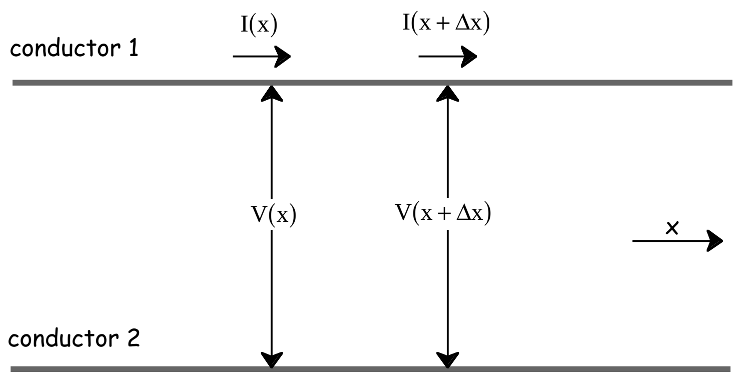


Figure 5: Propagation of voltage and current in a transmission line

Figure 5 shows the voltage and current at points x and x+Δx where x is the direction of propagation. The voltage drop from x to x+Δx is given by:

$$\Delta V = V(x + \Delta x) - V(x) = -L_o \Delta x \frac{\partial I}{\partial t} \qquad (7)$$

Re-arranging and taking the limit as $\Delta x \rightarrow 0$:

$$\frac{\partial V}{\partial x} = -L_o \frac{\partial I}{\partial t} \qquad (8)$$

The charge between x and x+Δx is $C_o \Delta x V$ and the time-rate change of this charge is $C_o \Delta x \cdot \partial V / \partial t$ but this time rate of change of charge only comes about due to the change in current from x to x+Δx so:

$$\Delta I = -C_o \Delta x \frac{\partial V}{\partial t} \qquad (9)$$

and again, taking the limit as $\Delta x \rightarrow 0$:

$$\frac{\partial I}{\partial x} = -C_o \frac{\partial V}{\partial t} \qquad (10)$$

We can take the partial of eq. (8) with respect to time and the partial of eq. (10) with respect to x and if the order in which the derivatives are taken does not matter then we have this wave equation for the current:

$$\frac{\partial^2 I}{\partial x^2} = C_o L_o \frac{\partial^2 I}{\partial t^2} \qquad (11)$$

Alternatively, taking the partial of eq. (8) with respect to x and the partial of eq. (10) with respect to time yields a wave equation for voltage:

$$\frac{\partial^2 V}{\partial x^2} = C_o L_o \frac{\partial^2 V}{\partial t^2} \qquad (12)$$

Any function of $V(x,t)$ where x and t appear in the following combination $x \pm vt$ will satisfy the wave equation (12) so long as:

$$v = \frac{1}{\sqrt{L_o C_o}}$$

In other words, $V(x,t) = f(x - vt)$ or $V(x,t) = g(x + vt)$ are solutions. The first corresponds to a wave propagating in the +x direction while the second corresponds to a wave propagating in the –x direction. Similarly the same requirement holds for $I(x,t)$.

Now consider the solutions corresponding to the wave propagating along the +x direction. We denote these by $V_+$ and $I_+$. Now consider eq. (8). Define $\eta = x - vt$ and with this, eq. (8) becomes:

$$\frac{\partial V_+}{\partial \eta}\frac{\partial \eta}{\partial x} = -L_o \frac{\partial I_+}{\partial \eta}\frac{\partial \eta}{\partial t}$$

$$\frac{\partial V_+}{\partial \eta} = vL_o \frac{\partial I_+}{\partial \eta} = \sqrt{\frac{L_o}{C_o}}\frac{\partial I_+}{\partial \eta}$$

Integrating with respect to $\eta$ we end up with

$$V_+ = +Z_o I_+ \qquad (13)$$

where

$$Z_o = \sqrt{\frac{L_o}{C_o}}$$

Following the same technique we get a similar expression for the wave propagating along the –x direction:

$$V_- = -Z_o I_- \qquad (14)$$

**Reflections**

Referring now to Fig. 3, if $R = Z_o$ then the relationship between current and voltage at point B is automatically satisfied for both the cable and the resistor. But if the impedance of the cable is not equal to the resistor, R, there will be a reflection to satisfy the condition that the incident plus reflected voltage V (and current I) satisfy V=IR.

We allow for a reflected voltage pulse that we assume is proportional to the incident pulse. Taking the incident pulse along the +x direction, we write:

$$V_- = \Gamma V_+ \qquad (15)$$

and the total current (I) through and the total voltage (V) across the resistor as:

$$V = V_+ + V_-$$
$$I = I_+ + I_-$$

Using eq. (15) and the relations (13) and (14):

$$V = (1+\Gamma)V_+$$
$$I = (1-\Gamma)I_+$$

$$R = \frac{V}{I} = \frac{1+\Gamma}{1-\Gamma} Z_O$$

Solving for the above:

$$\Gamma = \frac{R - Z_o}{R + Z_o} \qquad (16)$$

Now we consider three extreme cases:

1. $R = Z_o \Rightarrow \Gamma = 0$: In this case there is no reflected pulse.
2. $R = 0 \Rightarrow \Gamma = -1$: If there is a short circuit between the inner and outer conductor there is a reflected voltage pulse equal in magnitude to and opposite in sign guaranteeing that the voltage between the outer and inner conductor at point B is always zero.
3. $R = \infty \Rightarrow \Gamma = +1$: If there is an open circuit between the inner and outer conductor there is a reflected voltage pulse equal in magnitude to and of the same sign guaranteeing that the current between the outer and inner conductor at point B is always zero.