Maxwell's Equations

Honors Physics Note 006

Honors Physics P222 - Spring, 2004

Introduction

In this note I briefly review Maxwell's equations in their so-called *integral* form and their *differential* form. I strongly recommend references [1] and [2]. I also recommend that you go through this note carefully and if there is something you do not understand – please see me. First we start with the integral form.

Gauss' Law - Electric Field

Gauss' Law for the electric field is:

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{1}{\epsilon_0} \int_V \rho dV \tag{1}$$

The integral on the left hand side of the above equation is taken over a closed surface S that bounds the volume V over which the integral on the right hand side is performed. The volume charge density is given by ρ.

Gauss' Law - Magnetic Field

Gauss' Law for the magnetic field is:

$$\oint_S \vec{B} \cdot \hat{n} da = 0 \tag{2}$$

This follows from the apparent non-existence of magnetic monopoles.

Faraday's Law

Faraday's law states that a change in flux of magnetic field gives rise to an induced electric field:

$$\oint_C \vec{B} \cdot d\vec{\ell} = -\frac{\partial}{\partial t} \int_S \vec{E} \cdot \hat{n} da \tag{3}$$

The integral on the left hand side of the above equation is taken over a closed path C that bounds the surface S over which the integral on the right hand side is performed. The direction of the normal $\hat{n}$ to the surface S is related to the direction in which the closed path integral is taken by the right hand rule. Also note that there are an infinity of surfaces S bounded by the same closed path C. The surface integral is referred to as the *flux* of magnetic field and denoted by Φ_B.

Ampere's Law - and its Modification

Ampere's law gives us a method for calculating the magnetic field given a distribution of currents. But like with Gauss' law, its practical application requires some symmetry.

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \vec{J} \cdot \hat{n} da \tag{4}$$

The statements below equation 3 relating the surface of integration S and the closed path that bounds it C apply here as well. The vector $\vec{J}$ represents current per area.

As discussed in class and Sections 32-9 and 32-10 of your text, the above equation needs to be modified in that case where the flux of electric field, Φ_E, changes with time. The example of a spherically symmetric charge distribution leaking charge radially and symmetrically (recall the problem of Quiz 2) also support a change. The modified Ampere's law:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 \int_S \vec{J} \cdot \hat{n} da + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int_S \vec{E} \cdot \hat{n} da \tag{5}$$

Is it important to note that although there are an infinity of surfaces S bounded by the same closed path C, the two surface integrals on the right hand side of equation 5 must be taken over the same surface.

Conservation of Charge

To see how equation 5 implies conservation of charge imagine the closed path on the left hand side as a circle in the horizontal plane and the direction of the integral is counter-clockwise as viewed from above. Choose as a surface a hemisphere of the same radius as the circle and located above the horizontal plane. The circle bounds the hemisphere. In the application of equation 5 the surface normal on the hemisphere is directed out from the hemisphere. Call this *case I.*

Now reflect this arrangement about the horizontal plane. Viewed from below the circular path of integration is still counter-clockwise, the hemisphere is below the horizontal plane and call this *case II.* The circles of the two cases coincide.

Apply equation 5 to the two cases and add. On the left hand side we take identical line integrals in opposing direction and they add up to zero. When we add the right hand sides we have two integrals taken over the same *closed* surface. In other words:

$$\mu_0 \oint_S \vec{J} \cdot \hat{n} da + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \oint_S \vec{E} \cdot \hat{n} da = 0 \tag{6}$$

Now apply equation 1 to get:

$$\oint_S \vec{J} \cdot \hat{n} da = -\frac{\partial}{\partial t} \int_V \rho dV \tag{7}$$

But this is just a statement that the flux of current density over a closed surface equals the negative of the time rate of change of the total charge in the volume bounded by the closed surface. This is equivalent to a statement of the conservation of charge.

Gauss' Divergence Theorem and Stoke's Theorem

Gauss' Divergence Theorem and Stoke's Theorem and mathematical theorems. We covered the proof of the first in class and the proofs for both are given in Chapter 2 of reference [2], copies of which were distributed in class.

First the divergence theorem for any vector field $\vec{A}$:

$$\oint_S \vec{A} \cdot \hat{n} da = \int_V \vec{\nabla} \cdot \vec{A} dV \tag{8}$$

The integral on the left hand side of the above equation is taken over a closed surface S that bounds the volume V over which the integral on the right hand side is performed. $\vec{\nabla} \cdot \vec{A}$ is the *divergence* of $\vec{A}$.

Next the Stoke's theorem for any vector field $\vec{A}$:

$$\oint_C \vec{A} \cdot d\vec{\ell} = \int_S \vec{\nabla} \times \vec{A} \cdot \hat{n} da \tag{9}$$

The integral on the left hand side of the above equation is taken over a closed path C that bounds the surface S over which the integral on the right hand side is performed. $\vec{\nabla} \times \vec{A}$ is the *curl* of $\vec{A}$.

Differential Form of Maxwell's Equations

Applying the above two theorems (equations 8 and 9) to equations 1, 2, 3 and 5 results in the differential forms of Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{10}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{11}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{13}$$

And just for completeness we can also write the differential form of equation 7 that describes charge conservation as:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \tag{14}$$

Please make sure you understand how to arrive at these differential forms.

References

[1] R. P. Feynman, *The Feynman Lectures of Physics*, Volume II.

[2] E. M. Purcell, *Electricity and Magnetism*, Berkeley Physics Course Volume 2.