18–3 All of classical physics

In Table 18–1 we have all that was known of fundamental *classical* physics, that is, the physics that was known by 1905. Here it all is, in one table. With these equations we can understand the complete realm of classical physics.

First we have the Maxwell equations—written in both the expanded form and the short mathematical form. Then there is the conservation of charge, which is even written in parentheses, because the moment we have the complete Maxwell equations, we can deduce from them the conservation of charge. So the table is even a little redundant. Next, we have written the force law, because having all the electric and magnetic fields doesn't tell us anything until we know what they do to charges. Knowing $\boldsymbol{E}$ and $\boldsymbol{B}$, however, we can find the force on an object with the charge q moving with velocity $\boldsymbol{v}$. Finally, having the force doesn't tell us anything until we know what happens when a force pushes on something; we need the law of motion, which is that the force is equal to the rate of change of the momentum. (Remember? We had that in Volume I.) We even include relativity effects by writing the momentum as $p = m_0v/\sqrt{1 - v^2/c^2}$.

If we really want to be complete, we should add one more law—Newton's law of gravitation—so we put that at the end.

Therefore in one small table we have all the fundamental laws of classical physics—even with room to write them out in words and with some redundancy. This is a great moment. We have climbed a great peak. We are on the top of K-2—we are nearly ready for Mount Everest, which is quantum mechanics. We have climbed the peak of a "Great Divide," and now we can go down the other side.

We have mainly been trying to learn how to understand the equations. Now that we have the whole thing put together, we are going to study what the equations mean—what new things they say that we haven't already seen. We've been working hard to get up to this point. It has been a great effort, but now we are going to have nice coasting downhill as we see all the consequences of our accomplishment.

18–4 A travelling field

Now for the new consequences. They come from putting together all of Maxwell's equations. First, let's see what would happen in a circumstance which we pick to be particularly simple. By assuming that all the quantities vary only in one coordinate, we will have a one-dimensional problem. The situation is shown in Fig. 18–3. We have a sheet of charge located on the yz-plane. The sheet is first at rest, then instantaneously given a velocity u in the y-direction, and kept moving with this constant velocity. You might worry about having such an "infinite" acceleration, but it doesn't really matter; just imagine that the velocity is brought to u very quickly. So we have suddenly a surface current J (J is the current per unit

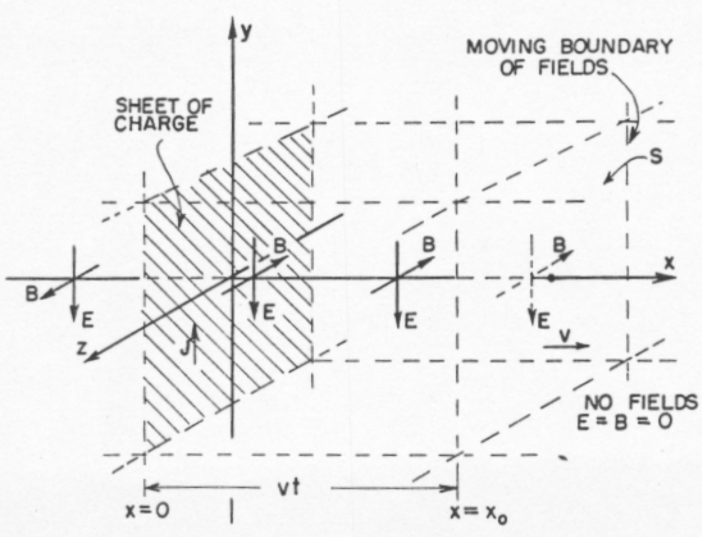

Fig. 18–3. An infinite sheet of charge is suddenly set into motion parallel to itself. There are magnetic and electric fields that propagate out from the sheet at a constant speed.

width in the z-direction). To keep the problem simple, we suppose that there is also a stationary sheet of charge of opposite sign superposed on the yz-plane, so that there are no electrostatic effects. Also, although in the figure we show only what is happening in a finite region, we imagine that the sheet extends to infinity in $\pm y$ and $\pm z$. In other words, we have a situation where there is no current, and then suddenly there is a uniform sheet of current. What will happen?

Well, when there is a sheet of current in the plus y-direction, there is, as we know, a magnetic field generated which will be in the minus z-direction for $x > 0$ and in the opposite direction for $x < 0$. We could find the magnitude of $\boldsymbol{B}$ by using the fact that the line integral of the magnetic field will be equal to the current over $\epsilon_0 c^2$. We would get that $B = J/2\epsilon_0 c^2$ (since the current I in a strip of width w is Jw and the line integral of $\boldsymbol{B}$ is $2Bw$).

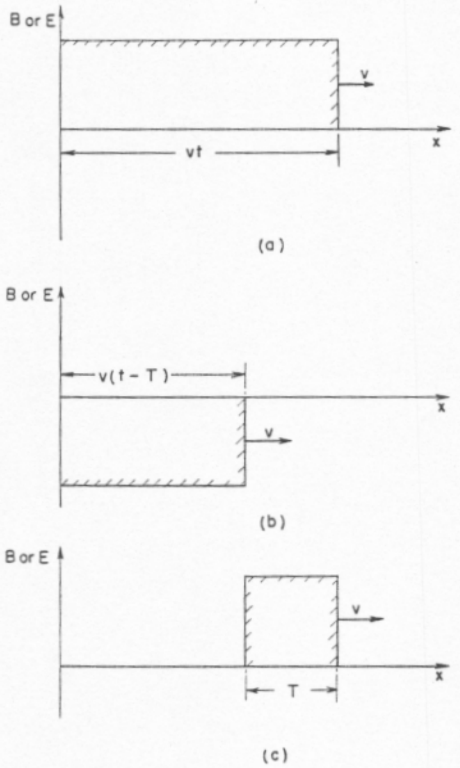

Fig. 18–4. (a) The magnitude of $\boldsymbol{B}$ (or $\boldsymbol{E}$) as a function of x at the time t after the charge sheet is set in motion. (b) The fields for a charge sheet set in motion, toward negative y at $t = T$. (c) The sum of (a) and (b).

This gives us the field next to the sheet—for small x—but since we are imagining an infinite sheet, we would expect the same argument to give the magnetic field farther out for larger values of x. However, that would mean that the moment we turn on the current, the magnetic field is suddenly changed from zero to a finite value everywhere. But wait! If the magnetic field is suddenly changed, it will produce tremendous electrical effects. (If it changes in *any* way, there are electrical effects.) So because we moved the sheet of charge, we make a changing magnetic field, and therefore electric fields must be generated. If there are electric fields generated, they had to start from zero and change to something else. There will be some $\partial \boldsymbol{E}/\partial t$ that will make a contribution, together with the current $\boldsymbol{J}$, to the production of the magnetic field. So through the various equations there is a big intermixing, and we have to try to solve for all the fields at once.

By looking at the Maxwell equations alone, it is not easy to see directly how to get the solution. So we will first show you what the answer is and then verify that it does indeed satisfy the equations. The answer is the following: The field $\boldsymbol{B}$ that we computed is, in fact, generated right next to the current sheet (for small x). It must be so, because if we make a tiny loop around the sheet, there is no room for any electric flux to go through it. But the field $\boldsymbol{B}$ out farther—for larger x—is, at first, zero. It stays zero for awhile, and then suddenly turns on. In short, we turn on the current and the magnetic field immediately next to it turns on to a constant value $\boldsymbol{B}$; then the turning on of $\boldsymbol{B}$ spreads out from the source region. After a certain time, there is a uniform magnetic field everywhere out to some value x, and then zero beyond. Because of the symmetry, it spreads in both the plus and minus x-directions.

The $\boldsymbol{E}$-field does the same thing. Before $t = 0$ (when we turn on the current), the field is zero everywhere. Then after the time t, both $\boldsymbol{E}$ and $\boldsymbol{B}$ are uniform out to the distance $x = vt$, and zero beyond. The fields make their way forward like a tidal wave, with a front moving at a uniform velocity which turns out to be c, but for a while we will just call it v. A graph of the magnitude of $\boldsymbol{E}$ or $\boldsymbol{B}$ versus x, as they appear at the time t, is shown in Fig. 18–4(a). Looking again at Fig. 18–3, at the time t, the region between $x = \pm vt$ is "filled" with the fields, but they have not yet reached beyond. We emphasize again that we are assuming that the current sheet and, therefore the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, extend infinitely far in both the y- and z-directions. (We cannot draw an infinite sheet, so we have shown only what happens in a finite area.)

We want now to analyze quantitatively what is happening. To do that, we want to look at two cross-sectional views, a top view looking down along the y-axis, as shown in Fig. 18–5, and a side view looking back along the z-axis, as shown in Fig. 18–6. Suppose we start with the side view. We see the charged sheet moving up; the magnetic field points into the page for $+x$, and out of the page for $-x$, and the electric field is downward everywhere—out to $x = \pm vt$.

Let's see if these fields are consistent with Maxwell's equations. Let's first draw one of those loops that we use to calculate a line integral, say the rectangle Γ_2 shown in Fig. 18–6. You notice that one side of the rectangle is in the region where there are fields, but one side is in the region the fields have still not reached. There is some magnetic flux through this loop. If it is changing, there should be an emf around it. If the wavefront is moving, we will have a changing magnetic

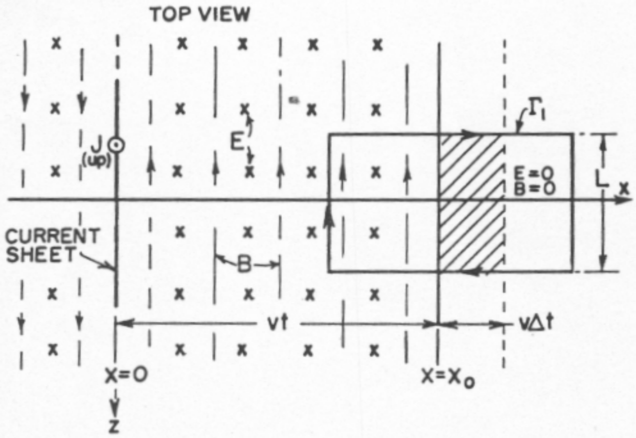

Fig. 18–5. Top view of Fig. 18–3.

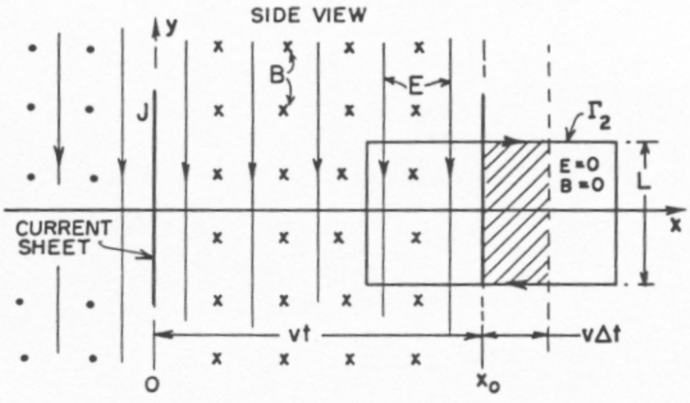

Fig. 18–6. Side view of Fig. 18–3.

flux, because the area in which $\boldsymbol{B}$ exists is progressively increasing at the velocity v. The flux inside Γ_2 is B times the part of the area inside Γ_2 which has a magnetic field. The rate of change of the flux, since the magnitude of $\boldsymbol{B}$ is constant, is the magnitude times the rate of change of the area. The rate of change of the area is easy. If the width of the rectangle Γ_2 is L, the area in which B exists changes by $Lv\,\Delta t$ in the time Δt. (See Fig. 18–6.) The rate of change of flux is then BLv. According to Faraday's law, this should equal the line integral of $\boldsymbol{E}$ around Γ_2, which is just EL. We have the equation

$$E = vB. \tag{18.10}$$

So if the ratio of E to B is v, the fields we have assumed will satisfy Faraday's equation.

But that is not the only equation; we have the other equation relating $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$c^2\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\epsilon_0} + \frac{\partial \boldsymbol{E}}{\partial t}. \tag{18.11}$$

To apply this equation, we look at the top view in Fig. 18–5. We have seen that this equation will give us the value of B next to the current sheet. Also, for any loop drawn outside the sheet but behind the wavefront, there is no curl of $\boldsymbol{B}$ nor any $\boldsymbol{j}$ or changing $\boldsymbol{E}$, so the equation is correct there. Now let's look at what happens for the curve Γ_1 that intersects the wavefront, as shown in Fig. 18–5. Here there are no currents, so Eq. (18.11) can be written—in integral form—as

$$c^2 \oint_{\Gamma_1} \boldsymbol{B} \cdot d\boldsymbol{s} = \frac{d}{dt} \int_{\text{inside } \Gamma_1} \boldsymbol{E} \cdot \boldsymbol{n}\, da. \tag{18.12}$$

The line integral of $\boldsymbol{B}$ is just B times L. The rate of change of the flux of $\boldsymbol{E}$ is due only to the advancing wavefront. The area inside Γ_1, where $\boldsymbol{E}$ is not zero, is increasing at the rate vL. The right-hand side of Eq. (18.12) is then vLE. That equation becomes

$$c^2B = Ev. \tag{18.13}$$

We have a solution in which we have a constant $\boldsymbol{B}$ and a constant $\boldsymbol{E}$ behind the front, both at right angles to the direction in which the front is moving and at right angles to each other. Maxwell's equations specify the ratio of E to B. From Eqs. (18.10) and (18.13),

$$E = vB, \qquad \text{and} \qquad E = \frac{c^2}{v}\,B.$$

But one moment! We have found *two different* conditions on the ratio E/B. Can such a field as we describe really exist? There is, of course, only one velocity v for which both of these equations can hold, namely $v = c$. The wavefront must travel with the velocity c. We have an example in which the electrical influence from a current propagates at a certain finite velocity c.

Now let's ask what happens if we suddenly stop the motion of the charged sheet after it has been on for a short time T. We can see what will happen by the principle of superposition. We had a current that was zero and then was suddenly turned on. We know the solution for that case. Now we are going to add another set of fields. We take another charged sheet and suddenly start it moving, in the opposite direction with the same speed, only at the time T after we started the first current. The total current of the two added together is first zero, then on for a time T, then off again—because the two currents cancel. We have a square "pulse" of current.

The new negative current produces the same fields as the positive one, only with all the signs reversed and, of course, delayed in time by T. A wavefront again travels out at the velocity c. At the time t it has reached the distance $x = \pm c(t - T)$, as shown in Fig. 18–4(b). So we have two "blocks" of field marching out at the speed c, as in parts (a) and (b) of Fig. 18–4. The combined fields are as shown in part (c) of the figure. The fields are zero for $x > ct$, they are constant (with the values we found above) between $x = c(t - T)$ and $x = ct$, and again zero for $x < c(t - T)$.

In short, we have a little piece of field—a block of thickness cT—which has left the current sheet and is travelling through space all by itself. The fields have "taken off"; they are propagating freely through space, no longer connected in any way with the source. The caterpillar has turned into a butterfly!

How can this bundle of electric and magnetic fields maintain itself? The answer is: by the combined effects of the Faraday law, $\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$, and the new term of Maxwell, $c^2\nabla \times \boldsymbol{B} = \partial \boldsymbol{E}/\partial t$. They cannot help maintaining themselves. Suppose the magnetic field were to disappear. There would be a changing magnetic field which would produce an electric field. If this electric field tries to go away, the changing electric field would create a magnetic field back again. So by a perpetual interplay—by the swishing back and forth from one field to the other—they must go on forever. It is impossible for them to disappear.* They maintain themselves in a kind of a dance—one making the other, the second making the first—propagating onward through space.

18–5 The speed of light

We have a wave which leaves the material source and goes outward at the velocity c, which is the speed of light. But let's go back a moment. From a historical point of view, it wasn't known that the coefficient c in Maxwell's equations was also the speed of light propagation. There was just a constant in the equations. We have called it c from the beginning, because we knew what it would turn out to be. We didn't think it would be sensible to make you learn the formulas with a different constant and then go back to substitute c wherever it belonged. From the point of view of electricity and magnetism, however, we just start out with two constants, ϵ_0 and c^2, that appear in the equations of electrostatics and magnetostatics:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \tag{18.14}$$

and

$$\nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\epsilon_0 c^2}\,. \tag{18.15}$$

If we take any *arbitrary* definition of a unit of charge, we can determine experimentally the constant ϵ_0 required in Eq. (18.14)—say by measuring the force between two unit charges at rest, using Coulomb's law. We must also determine experimentally the constant $\epsilon_0 c^2$ that appears in Eq. (18.15), which we can do, say, by measuring the force between two unit currents. (A unit current means one unit of charge per second.) The ratio of these two experimental constants is c^2—just another "electromagnetic constant."

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* Well, not quite. They can be "absorbed" if they get to a region where there are charges. By which we mean that other fields can be produced somewhere which superpose on these fields and "cancel" them by destructive interference (see Chapter 31, Vol. I).

Notice now that this constant c^2 is the same no matter what we choose for our unit of charge. If we put twice as much "charge"—say twice as many proton charges—in our "unit" of charge, ϵ_0 would need to be one-fourth as large. When we pass two of these "unit" currents through two wires, there will be twice as much "charge" per second in each wire, so the force between two wires is four times larger. The constant $\epsilon_0 c^2$ must be reduced by one-fourth. But the ratio $\epsilon_0 c^2/\epsilon_0$ is unchanged.

So just by experiments with charges and currents we find a number c^2 which turns out to be the square of the velocity of propagation of electromagnetic influences. From static measurements—by measuring the forces between two unit charges and between two unit currents—we find that $c = 3.00 \times 10^8$ meters/sec. When Maxwell first made this calculation with his equations, he said that bundles of electric and magnetic fields should be propagated at this speed. He also remarked on the mysterious coincidence that this was the same as the speed of light. "We can scarcely avoid the inference," said Maxwell, "that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena."

Maxwell had made one of the great unifications of physics. Before his time, there was light, and there was electricity and magnetism. The latter two had been unified by the experimental work of Faraday, Oersted, and Ampere. Then, all of a sudden, light was no longer "something else," but was only electricity and magnetism in this new form—little pieces of electric and magnetic fields which propagate through space on their own.

We have called your attention to some characteristics of this special solution, which turn out to be true, however, for *any* electromagnetic wave: that the magnetic field is perpendicular to the direction of motion of the wavefront; that the electric field is likewise perpendicular to the direction of motion of the wavefront; and that the two vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ are perpendicular to each other. Furthermore, the magnitude of the electric field E is equal to c times the magnitude of the magnetic field B. These three facts—that the two fields are transverse to the direction of propagation, that $\boldsymbol{B}$ is perpendicular to $\boldsymbol{E}$, and that $E = cB$—are generally true for any electromagnetic wave. Our special case is a good one—it shows all the main features of electromagnetic waves.

18–6 Solving Maxwell's equations; the potentials and the wave equation

Now we would like to do something mathematical; we want to write Maxwell's equations in a simpler form. You may consider that we are complicating them, but if you will be patient a little bit, they will suddenly come out simpler. Although by this time you are thoroughly used to each of the Maxwell equations, there are many pieces that must all be put together. That's what we want to do.

We begin with $\nabla \cdot \boldsymbol{B} = 0$—the simplest of the equations. We know that it implies that $\boldsymbol{B}$ is the curl of something. So, if we write

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{18.16}$$

we have already solved one of Maxwell's equations. (Incidentally, you appreciate that it remains true that another vector $\boldsymbol{A}'$ would be just as good if $\boldsymbol{A}' = \boldsymbol{A} + \nabla\psi$—where ψ is any scalar field—because the curl of $\nabla\psi$ is zero, and $\boldsymbol{B}$ is still the same. We have talked about that before.)

We take next the Faraday law, $\nabla \times \boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$, because it doesn't involve any currents or charges. If we write $\boldsymbol{B}$ as $\nabla \times \boldsymbol{A}$ and differentiate with respect to t, we can write Faraday's law in the form

$$\nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\nabla \times \boldsymbol{A}.$$

Since we can differentiate either with respect to time or to space first, we can also write this equation as

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0. \tag{18.17}$$