The Maxwell Equations

18–1 Maxwell's equations

In this chapter we come back to the complete set of the four Maxwell equations that we took as our starting point in Chapter 1. Until now, we have been studying Maxwell's equations in bits and pieces; it is time to add one final piece, and to put them all together. We will then have the complete and correct story for electromagnetic fields that may be changing with time in any way. Anything said in this chapter that contradicts something said earlier is true and what was said earlier is false—because what was said earlier applied to such special situations as, for instance, steady currents or fixed charges. Although we have been very careful to point out the restrictions whenever we wrote an equation, it is easy to forget all of the qualifications and to learn too well the wrong equations. Now we are ready to give the whole truth, with no qualifications (or almost none).

The complete Maxwell equations are written in Table 18–1, in words as well as in mathematical symbols. The fact that the words are equivalent to the equations should by this time be familiar—you should be able to translate back and forth from one form to the other.

The first equation—that the divergence of $\boldsymbol{E}$ is the charge density over ϵ_0—is true in general. In dynamic as well as in static fields, Gauss' law is always valid. The flux of $\boldsymbol{E}$ through any closed surface is proportional to the charge inside. The third equation is the corresponding general law for magnetic fields. Since there are no magnetic charges, the flux of $\boldsymbol{B}$ through any closed surface is always zero. The second equation, that the curl of $\boldsymbol{E}$ is $-\partial\boldsymbol{B}/\partial t$, is Faraday's law and was discussed in the last two chapters. It also is generally true. The last equation has something new. We have seen before only the part of it which holds for steady currents. In that case we said that the curl of $\boldsymbol{B}$ is $\boldsymbol{j}/\epsilon_0c^2$, but the correct general equation has a new part that was discovered by Maxwell.

Until Maxwell's work, the known laws of electricity and magnetism were those we have studied in Chapters 3 through 17. In particular, the equation for the magnetic field of steady currents was known only as

$$\nabla\times\boldsymbol{B}=\frac{\boldsymbol{j}}{\epsilon_0c^2}. \tag{18.1}$$

Maxwell began by considering these known laws and expressing them as differential equations, as we have done here. (Although the ∇ notation was not yet invented, it is mainly due to Maxwell that the importance of the combinations of derivatives, which we today call the curl and the divergence, first became apparent.) He then noticed that there was something strange about Eq. (18.1). If one takes the divergence of this equation, the left-hand side will be zero, because the divergence of a curl is always zero. So this equation requires that the divergence of $\boldsymbol{j}$ also be zero. But if the divergence of $\boldsymbol{j}$ is zero, then the total flux of current out of any closed surface is also zero.

The flux of current from a closed surface is the decrease of the charge inside the surface. This certainly cannot in general be zero because we know that the charges can be moved from one place to another. The equation

$$\nabla\cdot\boldsymbol{j}=-\frac{\partial\rho}{\partial t} \tag{18.2}$$

has, in fact, been almost our definition of $\boldsymbol{j}$. This equation expresses the very funda-

Table 18–1 Classical Physics

Maxwell's equations

I. $\nabla \cdot E = \dfrac{\rho}{\epsilon_0}$ (Flux of E through a closed surface) = (Charge inside)/ϵ_0

II. $\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Line integral of E around a loop) $= -\dfrac{d}{dt}$ (Flux of B through the loop)

III. $\nabla \cdot B = 0$ (Flux of B through a closed surface) = 0

IV. $c^2 \nabla \times B = \dfrac{j}{\epsilon_0} + \dfrac{\partial E}{\partial t}$ c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+ \dfrac{\partial}{\partial t}$ (Flux of E through the loop)

[Conservation of charge

$\nabla \cdot j = -\dfrac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) $= -\dfrac{\partial}{\partial t}$ (Charge inside)]

Force law

$$F = q(E + v \times B)$$

Law of motion

$\dfrac{d}{dt}(p) = F$, where $p = \dfrac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

$$F = -G \frac{m_1 m_2}{r^2} e_r$$

mental law that electric charge is conserved—any flow of charge must come from some supply. Maxwell appreciated this difficulty and proposed that it could be avoided by adding the term $\partial E/\partial t$ to the right-hand side of Eq. (18.1); he then got the fourth equation in Table 18–1:

$$\text{IV.} \qquad c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}.$$

It was not yet customary in Maxwell's time to think in terms of abstract fields. Maxwell discussed his ideas in terms of a model in which the vacuum was like an elastic solid. He also tried to explain the meaning of his new equation in terms of the mechanical model. There was much reluctance to accept his theory, first because of the model, and second because there was at first no experimental justification. Today, we understand better that what counts are the equations themselves and not the model used to get them. We may only question whether the equations are true or false. This is answered by doing experiments, and untold numbers of experiments have confirmed Maxwell's equations. If we take away the scaffolding he used to build it, we find that Maxwell's beautiful edifice stands on its own. He brought together all of the laws of electricity and magnetism and made one complete and beautiful theory.

Let us show that the extra term is just what is required to straighten out the difficulty Maxwell discovered. Taking the divergence of his equation (IV in Table 18–1), we must have that the divergence of the right-hand side is zero:

$$\nabla \cdot \frac{j}{\epsilon_0} + \nabla \cdot \frac{\partial E}{\partial t} = 0. \tag{18.3}$$

In the second term, the order of the derivatives with respect to coordinates and time can be reversed, so the equation can be rewritten as

$$\nabla \cdot \boldsymbol{j} + \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{E} = 0. \tag{18.4}$$

But the first of Maxwell's equations says that the divergence of $\boldsymbol{E}$ is ρ/ϵ_0. Inserting this equality in Eq. (18.4), we get back Eq. (18.2), which we know is true. Conversely, if we accept Maxwell's equations—and we do because no one has ever found an experiment that disagrees with them—we must conclude that charge is always conserved.

The laws of physics have no answer to the question: "What happens if a charge is suddenly created at this point—what electromagnetic effects are produced?" No answer can be given because our equations say it doesn't happen. If it *were* to happen, we would need new laws, but we cannot say what they would be. We have not had the chance to observe how a world without charge conservation behaves. According to our equations, if you suddenly place a charge at some point, you had to carry it there from somewhere else. In that case, we can say what would happen.

When we added a new term to the equation for the curl of $\boldsymbol{E}$, we found that a whole new class of phenomena was described. We shall see that Maxwell's little addition to the equation for $\nabla \times \boldsymbol{B}$ also has far-reaching consequences. We can touch on only a few of them in this chapter.

18–2 How the new term works

As our first example we consider what happens with a spherically symmetric radial distribution of current. Suppose we imagine a little sphere with radioactive material on it. This radioactive material is squirting out some charged particles. (Or we could imagine a large block of jello with a small hole in the center into which some charge had been injected with a hypodermic needle and from which the charge is slowly leaking out.) In either case we would have a current that is everywhere radially outward. We will assume that it has the same magnitude in all directions.

Let the total charge inside any radius r be $Q(r)$. If the radial current density at the same radius is $\boldsymbol{j}(r)$, then Eq. (18.2) requires that Q decreases at the rate

$$\frac{\partial Q(r)}{\partial t} = -4\pi r^2 j(r). \tag{18.5}$$

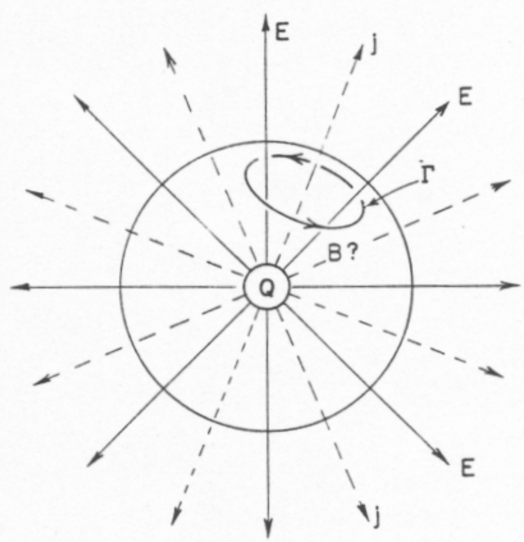

Fig. 18–1. What is the magnetic field of a spherically symmetric current?

We now ask about the magnetic field produced by the currents in this situation. Suppose we draw some loop Γ on a sphere of radius r, as shown in Fig. 18–1. There is some current through this loop, so we might expect to find a magnetic field circulating in the direction shown.

But we are already in difficulty. How can the $\boldsymbol{B}$ have any particular direction on the sphere? A different choice of Γ would allow us to conclude that its direction is exactly opposite to that shown. So how *can* there be any circulation of $\boldsymbol{B}$ around the currents?

We are saved by Maxwell's equation. The circulation of $\boldsymbol{B}$ depends not only on the total *current* through Γ but also on the rate of change with time of the *electric flux* through it. It must be that these two parts just cancel. Let's see if that works out.

The electric field at the radius r must be $Q(r)/4\pi\epsilon_0 r^2$—so long as the charge is symmetrically distributed, as we assume. It is radial, and its rate of change is then

$$\frac{\partial E}{\partial t} = \frac{1}{4\pi\epsilon_0 r^2} \frac{\partial Q}{\partial t}. \tag{18.6}$$

Comparing this with Eq. (18.5), we see that at any radius

$$\frac{\partial E}{\partial t} = -\frac{j}{\epsilon_0}. \tag{18.7}$$

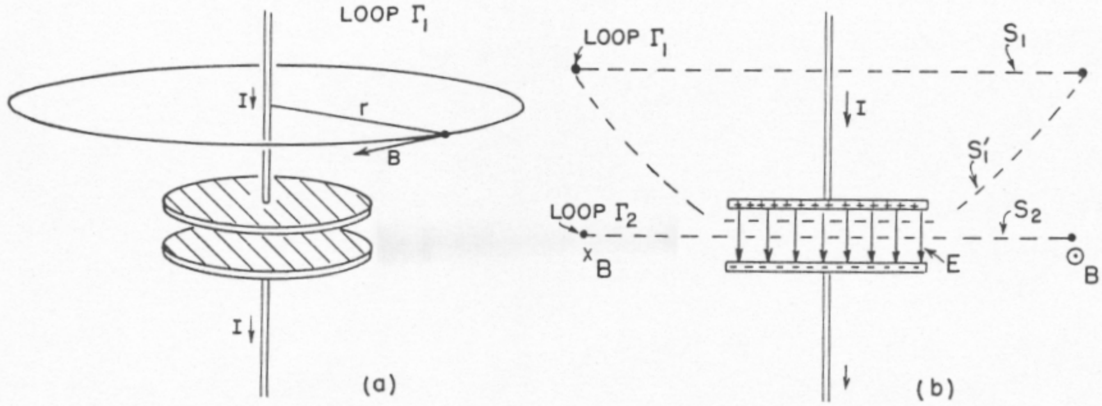

Fig. 18–2. The magnetic field near a charging capacitor.

In Eq. IV the two source terms cancel and the curl of $\boldsymbol{B}$ is always zero. There is no magnetic field in our example.

As our second example, we consider the magnetic field of a wire used to charge a parallel-plate condenser (see Fig. 18–2). If the charge Q on the plates is changing with time (but not too fast), the current in the wires is equal to dQ/dt. We would expect that this current will produce a magnetic field that encircles the wire. Surely, the current close to the wire must produce the normal magnetic field—it cannot depend on where the current is going.

Suppose we take a loop Γ_1 which is a circle with radius r, as shown in part (a) of the figure. The line integral of the magnetic field should be equal to the current I divided by $\epsilon_0 c^2$. We have

$$2\pi r B = \frac{I}{\epsilon_0 c^2}. \tag{18.8}$$

This is what we would get for a steady current, but it is also correct with Maxwell's addition, because if we consider the plane surface S inside the circle, there are no electric fields on it (assuming the wire to be a very good conductor). The surface integral of $\partial E/\partial t$ is zero.

Suppose, however, that we now slowly move the curve Γ downward. We get always the same result until we draw even with the plates of the condenser. Then the current I goes to zero. Does the magnetic field disappear? That would be quite strange. Let's see what Maxwell's equation says for the curve Γ_2, which is a circle of radius r whose plane passes between the condenser plates [Fig. 18–2(b)]. The line integral of $\boldsymbol{B}$ around Γ_2 is $2\pi r B$. This must equal the time derivative of the flux of $\boldsymbol{E}$ through the plane circular surface S_2. This flux of $\boldsymbol{E}$, we know from Gauss' law, must be equal to $1/\epsilon_0$ times the charge Q on one of the condenser plates. We have

$$c^2\, 2\pi r B = \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right). \tag{18.9}$$

That is very convenient. It is the same result we found in Eq. (18.8). Integrating over the changing electric field gives the same magnetic field as does integrating over the current in the wire. Of course, that is just what Maxwell's equation says. It is easy to see that this must always be so by applying our same arguments to the two surfaces S_1 and S_1' that are bounded by the same circle Γ_1 in Fig. 18–2(b). Through S_1 there is the current I, but no electric flux. Through S_1' there is no current, but an electric flux changing at the rate I/ϵ_0. The same $\boldsymbol{B}$ is obtained if we use Eq. IV with either surface.

From our discussion so far of Maxwell's new term, you may have the impression that it doesn't add much—that it just fixes up the equations to agree with what we already expect. It is true that if we just consider Eq. IV *by itself*, nothing particularly new comes out. The words "*by itself*" are, however, all-important. Maxwell's small change in Eq. IV, when *combined with the other* equations, does indeed produce much that is new and important. Before we take up these matters, however, we want to speak more about Table 18–1.