

of the fields depend only upon x —that there are no *variations* of the fields with y and z . We are, of course, considering plane waves again. We should expect to get results something like those in the previous section. In fact, we will find precisely the same answers. You may ask: “Why do it all over again?” It is important to do it again, first, because we did not show that the waves we found were the most general solutions for plane waves, and second, because we found the fields only from a very particular kind of current source. We would like to ask now: What is the most general kind of one-dimensional wave there can be in free space? We cannot find that by seeing what happens for this or that particular source, but must work with greater generality. Also we are going to work this time with differential equations instead of with integral forms. Although we will get the same results, it is a way of practicing back and forth to show that it doesn't make any difference which way you go. You should know how to do things every which way, because when you get a hard problem, you will often find that only one of the various ways is tractable.

We could consider directly the solution of the wave equation for some electromagnetic quantity. Instead, we want to start right from the beginning with Maxwell's equations in free space so that you can see their close relationship to the electromagnetic waves. So we start with the equations in (20.1), setting the charges and currents equal to zero. They become

$$\begin{aligned}
 \text{I. } \quad \nabla \cdot \mathbf{E} &= 0 \\
 \text{II. } \quad \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
 \text{III. } \quad \nabla \cdot \mathbf{B} &= 0 \\
 \text{IV. } \quad c^2 \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t}
 \end{aligned}
 \tag{20.12}$$

We write the first equation out in components:

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0.
 \tag{20.13}$$

We are assuming that there are no variations with y and z , so the last two terms are zero. This equation then tells us that

$$\frac{\partial E_x}{\partial x} = 0.
 \tag{20.14}$$

Its solution is that E_x , the component of the electric field in the x -direction, is a constant in space. If you look at IV in (20.12), supposing no \mathbf{B} -variation in y and z either, you can see that E_x is also constant in time. Such a field could be the steady DC field from some charged condenser plates a long distance away. We are not interested now in such an uninteresting static field; we are at the moment interested only in dynamically varying fields. For *dynamic* fields, $E_x = 0$.

We have then the important result that for the propagation of plane waves in any direction, *the electric field must be at right angles to the direction of propagation*. It can, of course, still vary in a complicated way with the coordinate x .

The transverse \mathbf{E} -field can always be resolved into two components, say the y -component and the z -component. So let's first work out a case in which the electric field has only one transverse component. We'll take first an electric field that is always in the y -direction, with zero z -component. Evidently, if we solve this problem we can also solve for the case where the electric field is always in the z -direction. The general solution can always be expressed as the superposition of two such fields.

How easy our equations now get. The only component of the electric field that is not zero is E_y , and all derivatives—except those with respect to x —are zero. The rest of Maxwell's equations then become quite simple.

Let's look next at the second of Maxwell's equations [III of Eq. (20.12)]. Writing out the components of the curl E , we have

$$\begin{aligned}(\nabla \times E)_x &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0, \\(\nabla \times E)_y &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0, \\(\nabla \times E)_z &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}.\end{aligned}$$

The x -component of $\nabla \times E$ is zero because the derivatives with respect to y and z are zero. The y -component is also zero; the first term is zero because the derivative with respect to z is zero, and the second term is zero because E_z is zero. The only components of the curl of E that is not zero is the z -component, which is equal to $\partial E_y/\partial x$. Setting the three components of $\nabla \times E$ equal to the corresponding components of $-\partial B/\partial t$, we can conclude the following:

$$\frac{\partial B_x}{\partial t} = 0, \quad \frac{\partial B_y}{\partial t} = 0. \quad (20.15)$$

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x}. \quad (20.16)$$

Since the x -component of the magnetic field and the y -component of the magnetic field both have zero time derivatives, these two components are just constant fields and correspond to the magnetostatic solutions we found earlier. Somebody may have left some permanent magnets near where the waves are propagating. We will ignore these constant fields and set B_x and B_y equal to zero.

Incidentally, we would already have concluded that the x -component of B should be zero for a different reason. Since the divergence of B is zero (from the third Maxwell equation), applying the same arguments we used above for the electric field, we would conclude that the longitudinal component of the magnetic field can have no variation with x . Since we are ignoring such uniform fields in our wave solutions, we would have set B_x equal to zero. In plane electromagnetic waves the B -field, as well as the E -field, must be directed at right angles to the direction of propagation.

Equation (20.16) gives us the additional proposition that if the electric field has only a y -component, the magnetic field will have only a z -component. So E and B are at right angles to each other. This is exactly what happened in the special wave we have already considered.

We are now ready to use the last of Maxwell's equations for free space [IV of Eq. (20.12)]. Writing out the components, we have

$$\begin{aligned}c^2(\nabla \times B)_x &= c^2 \frac{\partial B_z}{\partial y} - c^2 \frac{\partial B_y}{\partial z} = \frac{\partial E_x}{\partial t}, \\c^2(\nabla \times B)_y &= c^2 \frac{\partial B_x}{\partial z} - c^2 \frac{\partial B_z}{\partial x} = \frac{\partial E_y}{\partial t}, \\c^2(\nabla \times B)_z &= c^2 \frac{\partial B_y}{\partial x} - c^2 \frac{\partial B_x}{\partial y} = \frac{\partial E_z}{\partial t}.\end{aligned} \quad (20.17)$$

Of the six derivatives of the components of B , only the term $\partial B_z/\partial x$ is not equal to zero. So the three equations give us simply

$$-c^2 \frac{\partial B_z}{\partial x} = \frac{\partial E_y}{\partial t}. \quad (20.18)$$

The result of all our work is that only one component each of the electric and magnetic fields is not zero, and that these components must satisfy Eqs. (20.16) and (20.18). The two equations can be combined into one if we differentiate the first with respect to x and the second with respect to t ; the left-hand sides of the

two equations will then be the same (except for the factor c^2). So we find that E_y satisfies the equation

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} = 0. \quad (20.19)$$

We have seen the same differential equation before, when we studied the propagation of sound. It is the wave equation for one-dimensional waves.

You should note that in the process of our derivation we have found something *more* than is contained in Eq. (20.11). Maxwell's equations have given us the further information that electromagnetic waves have field components only at right angles to the direction of the wave propagation.

Let's review what we know about the solutions of the one-dimensional wave equation. If any quantity ψ satisfies the one-dimensional wave equation

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0, \quad (20.20)$$

then one possible solution is a function $\psi(x, t)$ of the form

$$\psi(x, t) = f(x - ct), \quad (20.21)$$

that is, some function of the *single* variable $(x - ct)$. The function $f(x - ct)$ represents a "rigid" pattern in x which travels toward positive x at the speed c (see Fig. 20-4). For example, if the function f has a maximum when its argument is zero, then for $t = 0$ the maximum of ψ will occur at $x = 0$. At some later time, say $t = 10$, ψ will have its maximum at $x = 10c$. As time goes on, the maximum moves toward positive x at the speed c .

Sometimes it is more convenient to say that a solution of the one-dimensional wave equation is a function of $(t - x/c)$. However, this is saying the same thing, because any function of $(t - x/c)$ is also a function of $(x - ct)$:

$$F(t - x/c) = F\left[-\frac{x - ct}{c}\right] = f(x - ct).$$

Let's show that $f(x - ct)$ is indeed a solution of the wave equation. Since it is a function of only one variable—the variable $(x - ct)$ —we will let f' represent the derivative of f with respect to its variable and f'' represent the second derivative of f . Differentiating Eq. (20.21) with respect to x , we have

$$\frac{\partial \psi}{\partial x} = f'(x - ct),$$

since the derivative of $(x - ct)$ with respect to x is 1. The second derivative of ψ with respect to x is clearly

$$\frac{\partial^2 \psi}{\partial x^2} = f''(x - ct). \quad (20.22)$$

Taking derivatives of ψ with respect to t , we find

$$\frac{\partial \psi}{\partial t} = f'(x - ct)(-c),$$

$$\frac{\partial^2 \psi}{\partial t^2} = +c^2 f''(x - ct). \quad (20.23)$$

We see that ψ does indeed satisfy the one-dimensional wave equation.

You may be wondering: "If I have the wave equation, how do I know that I should take $f(x - ct)$ as a solution? I don't like this backward method. Isn't there some *forward* way to find the solution?" Well, one good forward way is to know the solution. It is possible to "cook up" an apparently forward mathematical argument, especially because we know what the solution is supposed to be, but with an equation as simple as this we don't have to play games. Soon you will get so that when you see Eq. (20.20), you nearly simultaneously see

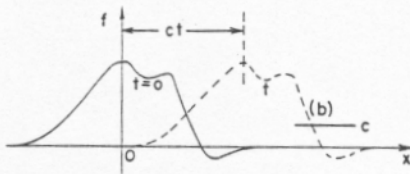


Fig. 20-4. The function $f(x - ct)$ represents a constant "shape" that travels toward positive x with the speed c .

$\psi = f(x - ct)$ as a solution. (Just as now when you see the integral of $x^2 dx$, you know right away that the answer is $x^3/3$.)

Actually you should also see a little more. Not only is any function of $(x - ct)$ a solution, but any function of $(x + ct)$ is also a solution. Since the wave equation contains only c^2 , changing the sign of c makes no difference. In fact, the *most general* solution of the one-dimensional wave equation is the sum of two arbitrary functions, one of $(x - ct)$ and the other of $(x + ct)$:

$$\psi = f(x - ct) + g(x + ct). \quad (20.24)$$

The first term represents a wave travelling toward positive x , and the second term an arbitrary wave travelling toward negative x . The general solution is the superposition of two such waves both existing at the same time.

We will leave the following amusing question for you to think about. Take a function ψ of the following form:

$$\psi = \cos kx \cos kct.$$

This equation isn't in the form of a function of $(x - ct)$ or of $(x + ct)$. Yet you can easily show that this function is a solution of the wave equation by direct substitution into Eq. (20.20). How can we then say that the general solution is of the form of Eq. (20.24)?

Applying our conclusions about the solution of the wave equation to the y -component of the electric field, E_y , we conclude that E_y can vary with x in any arbitrary fashion. However, the fields which do exist can always be considered as the sum of two patterns. One wave is sailing through space in one direction with speed c , with an associated magnetic field perpendicular to the electric field; another wave is travelling in the opposite direction with the same speed. Such waves correspond to the electromagnetic waves that we know about—light, radio-waves, infrared radiation, ultraviolet radiation, x-rays, and so on. We have already discussed the radiation of light in great detail in Vol. I. Since everything we learned there applies to any electromagnetic wave, we don't need to consider in great detail here the behavior of these waves.

We should perhaps make a few further remarks on the question of the polarization of the electromagnetic waves. In our solution we chose to consider the special case in which the electric field has only a y -component. There is clearly another solution for waves travelling in the plus or minus x -direction, with an electric field which has only a z -component. Since Maxwell's equations are linear, the general solution for one-dimensional waves propagating in the x -direction is the sum of waves of E_y and waves of E_z . This general solution is summarized in the following equations:

$$\begin{aligned} E &= (0, E_y, E_z) \\ E_y &= f(x - ct) + g(x + ct) \\ E_z &= F(x - ct) + G(x + ct) \\ B &= (0, B_y, B_z) \\ cB_z &= f(x - ct) - g(x + ct) \\ cB_y &= -F(x - ct) + G(x + ct). \end{aligned} \quad (20.25)$$

Such electromagnetic waves have an E -vector whose direction is not constant but which gyrates around in some arbitrary way in the yz -plane. At every point the magnetic field is always perpendicular to the electric field and to the direction of propagation.

If there are only waves travelling in one direction, say the positive x -direction, there is a simple rule which tells the relative orientation of the electric and magnetic fields. The rule is that the cross product $\mathbf{E} \times \mathbf{B}$ —which is, of course, a vector at right angles to both \mathbf{E} and \mathbf{B} —points in the direction in which the wave is travelling. If \mathbf{E} is rotated into \mathbf{B} by a right-hand screw, the screw points in the direction of the wave velocity. (We shall see later that the vector $\mathbf{E} \times \mathbf{B}$ has a special physical significance: it is a vector which describes the flow of energy in an electromagnetic field.)

20-2 Three-dimensional waves

We want now to turn to the subject of three-dimensional waves. We have already seen that the vector \mathbf{E} satisfies the wave equation. It is also easy to arrive at the same conclusion by arguing directly from Maxwell's equations. Suppose we start with the equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and take the curl of both sides:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}). \quad (20.26)$$

You will remember that the curl of the curl of any vector can be written as the sum of two terms, one involving the divergence and the other the Laplacian,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}.$$

In free space, however, the divergence of \mathbf{E} is zero, so only the Laplacian term remains. Also, from the fourth of Maxwell's equations in free space [Eq. (20.12)] the time derivative of $c^2 \nabla \times \mathbf{B}$ is the second derivative of \mathbf{E} with respect to t :

$$c^2 \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Equation (20.26) then becomes

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

which is the three-dimensional wave equation. Written out in all its glory, this equation is, of course,

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \quad (20.27)$$

How shall we find the general wave solution? The answer is that all the solutions of the three-dimensional wave equation can be represented as a superposition of the one-dimensional solutions we have already found. We obtained the equation for waves which move in the x -direction by supposing that the field did not depend on y and z . Obviously, there are other solutions in which the fields do not depend on x and z , representing waves going in the y -direction. Then there are solutions which do not depend on x and y , representing waves travelling in the z -direction. Or in general, since we have written our equations in vector form, the three-dimensional wave equation can have solutions which are plane waves moving in any direction at all. Again, since the equations are linear, we may have simultaneously as many plane waves as we wish, travelling in as many different directions. Thus the most general solution of the three-dimensional wave equation is a superposition of all sorts of plane waves moving in all sorts of directions.

Try to imagine what the electric and magnetic fields look like at present in the space in this lecture room. First of all, there is a steady magnetic field; it comes from the currents in the interior of the earth—that is, the earth's steady magnetic field. Then there are some irregular, nearly static electric fields produced perhaps by electric charges generated by friction as various people move about in their

chairs and rub their coat sleeves against the chair arms. Then there are other magnetic fields produced by oscillating currents in the electrical wiring—fields which vary at a frequency of 60 cycles per second, in synchronism with the generator at Boulder Dam. But more interesting are the electric and magnetic fields varying at much higher frequencies. For instance, as light travels from window to floor and wall to wall, there are little wiggles of the electric and magnetic fields moving along at 186,000 miles per second. Then there are also infrared waves travelling from the warm foreheads to the cold blackboard. And we have forgotten the ultraviolet light, the x-rays, and the radiowaves travelling through the room.

Flying across the room are electromagnetic waves which carry music of a jazz band. There are waves modulated by a series of impulses representing pictures of events going on in other parts of the world, or of imaginary aspirins dissolving in imaginary stomachs. To demonstrate the reality of these waves it is only necessary to turn on electronic equipment that converts these waves into pictures and sounds.

If we go into further detail to analyze even the smallest wiggles, there are tiny electromagnetic waves that have come into the room from enormous distances. There are now tiny oscillations of the electric field, whose crests are separated by a distance of one foot, that have come from millions of miles away, transmitted to the earth from the Mariner II space craft which has just passed Venus. Its signals carry summaries of information it has picked up about the planets (information obtained from electromagnetic waves that travelled from the planet to the space craft).

There are very tiny wiggles of the electric and magnetic fields that are waves which originated billions of light years away—from galaxies in the remotest corners of the universe. That this is true has been found by “filling the room with wires”—by building antennas as large as this room. Such radiowaves have been detected from places in space beyond the range of the greatest optical telescopes. Even they, the optical telescopes, are simply gatherers of electromagnetic waves. What we call the stars are only inferences, inferences drawn from the only physical reality we have yet gotten from them—from a careful study of the unendingly complex undulations of the electric and magnetic fields reaching us on earth.

There is, of course, more: the fields produced by lightning miles away, the fields of the charged cosmic ray particles as they zip through the room, and more, and more. What a complicated thing is the electric field in the space around you! Yet it always satisfies the three-dimensional wave equation.

20-3 Scientific imagination

I have asked you to imagine these electric and magnetic fields. What do you do? Do you know how? How do I imagine the electric and magnetic field? What do I actually see? What are the demands of scientific imagination? Is it any different from trying to imagine that the room is full of invisible angels? No, it is not like imagining invisible angels. It requires a much higher degree of imagination to understand the electromagnetic field than to understand invisible angels. Why? Because to make invisible angels understandable, all I have to do is to alter their properties *a little bit*—I make them slightly visible, and then I can see the shapes of their wings, and bodies, and halos. Once I succeed in imagining a visible angel, the abstraction required—which is to take almost invisible angels and imagine them completely invisible—is relatively easy. So you say, “Professor, please give me an approximate description of the electromagnetic waves, even though it may be slightly inaccurate, so that I too can see them as well as I can see almost invisible angels. Then I will modify the picture to the necessary abstraction.”

I'm sorry I can't do that for you. I don't know how. I have no picture of this electromagnetic field that is in any sense accurate. I have known about the electromagnetic field a long time—I was in the same position 25 years ago that you are now, and I have had 25 years more of experience thinking about these wiggling waves. When I start describing the magnetic field moving through space, I speak of the *E*- and *B* fields and wave my arms and you may imagine that I can see them.