Motion in Crossed E and B Fields

Honor Physics Note 006

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Introduction

Suppose we have a region of space with a constant magnetic field $\vec{B}$ oriented along the x-axis and a constant electric field $\vec{E}$ oriented along the z-axis. A positively charged particle of mass m and charge q is held at rest at the origin of coordinates and then released at $t = 0$. The particle will move along the trajectory shown in Figure 1.

Before we solve analytically for the motion of the particle we will first look at the situation qualitatively. Initially the particle is accelerated up along the positive z-axis because of a force due to the electric field, $\vec{F}_E = q\vec{E}$. And as the velocity v increases the particle also experiences an additional force due to the magnetic field given by $\vec{F}_B = q\vec{v} \times \vec{B}$. The particle trajectory curves in the $y - z$ plane and as the velocity increases due to the electric field so does the force due to the magnetic field. From the point of view of energy conservation, as the positively charged particle moves up along the $\vec{E}$ field the potential energy decreases linearly with distance along the z-axis and the kinetic energy increases accordingly. [This is analogous to the situation of dropping a mass from rest in the uniform gravitational field.]

Eventually the particle starts moving downward, anti-parallel to the $\vec{E}$ field and it loses kinetic energy. The particle is decelerated by the $\vec{E}$ field. Invoking energy conservation we see that the particle comes to rest when $z = 0$. But now we have the same conditions that started the motion to begin with – so the process keeps repeating. As time goes on the motion of the particle is constrained to within some range of z values but there is net motion along the $+y$ direction. The net motion is along the $+y$ direction, neither along the $\vec{E}$ field nor $\vec{B}$ field. In analyzing the motion using energy conservation, recall that only the $\vec{E}$ field and not the $\vec{B}$ can do work on the particle changing its kinetic energy.

If the charged particle were negatively charged, with the fields oriented as before, the net motion of the particle would still be along the $+y$ direction even though the trajectory as shown is reflected about the y-axis.

Analytical treatment

From the previous discussion we know that the velocity of the particle is constrained in the $y - z$ plane since there are no forces in the x direction. So we write the velocity vector as $\vec{v} = \dot{y}\hat{\mathbf{j}} + \dot{z}\hat{\mathbf{k}}$. We are using the convention for derivatives such that $\dot{y} = dy/dt$ and $\ddot{y} = d^2y/dt^2$. We are assuming that $\vec{B} = B\hat{\mathbf{i}}$ and $\vec{E} = E\hat{\mathbf{k}}$ and clearly $\vec{F}_E = qE\hat{\mathbf{k}}$. For the magnetic force:

$$\vec{F}_B = q\vec{v} \times \vec{B} = qB\dot{z}\hat{\mathbf{j}} - qB\dot{y}\hat{\mathbf{k}} \tag{1}$$

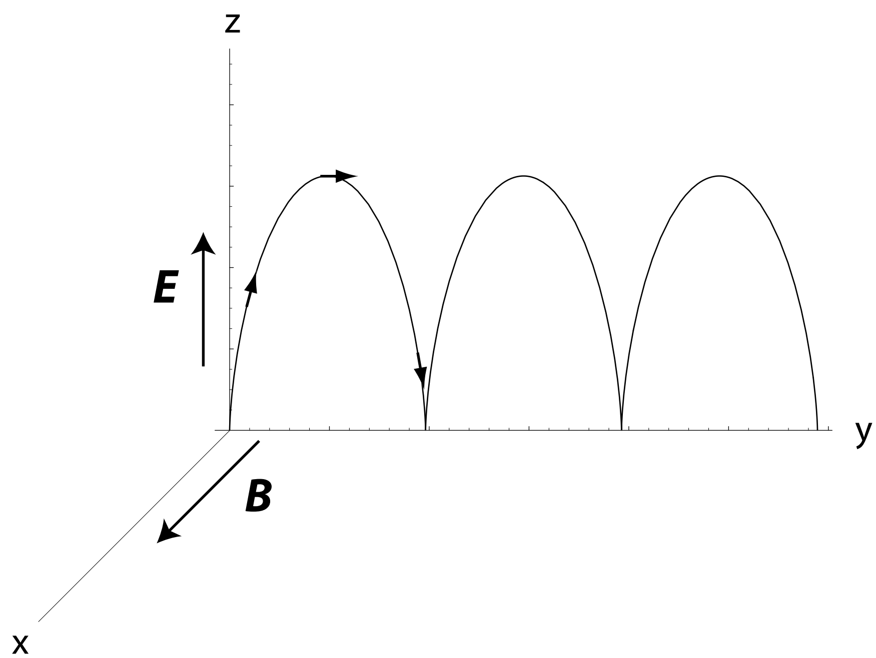

Figure 1: Motion

and for the total force:

$$\vec{F} = m\vec{a} = m(\ddot{y}\hat{\mathbf{j}} + \ddot{z}\hat{\mathbf{k}}) = qB\dot{z}\hat{\mathbf{j}} + q(E - B\dot{y})\hat{\mathbf{k}} \tag{2}$$

This implies two equations:

$$\ddot{y} = \frac{qB}{m}\dot{z} = \omega\dot{z} \tag{3}$$

and

$$\ddot{z} = \frac{qB}{m}\left(\frac{E}{B} - \dot{y}\right) = \omega\left(\frac{E}{B} - \dot{y}\right) \tag{4}$$

where in both of the above we let $\omega = qB/m$ often referred to as the *cyclotron* frequency. Equations 3 and 4 are said to be *coupled* in the sense that the z motion depends on y and vice-versa. One way to decouple these equations is to differentiate equation 3 with respect to time and then use this in 4 to eliminate z. But we'll just write down the solutions and let you confirm that they satisfy equations 3 and 4.

Here are the solutions:

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{E}{B} t + C_3 \tag{5}$$

$$z(t) = C_2 \cos \omega t - C_1 \sin \omega t + C_4 \tag{6}$$

where the four C's are constants. The constants are fixed by the initial conditions which are that the particle starts from rest at the origin. In other words $y(0) = z(0) = 0$ and $\dot{y}(0) = \dot{z}(0) = 0$. Imposing these on equations 5 and 6 we get:

$$y(t) = \frac{E}{\omega B}(\omega t - \sin \omega t) \tag{7}$$

$$z(t) = \frac{E}{\omega B}(1 - \cos \omega t) \tag{8}$$

Let's define $R = E/\omega B$ and then rewrite equations 7 and 8 in such a way to exploit $\cos^2 \omega t + \sin^2 \omega t = 1$. Here is what you get:

$$(y - R\omega t)^2 + (z - R)^2 = R^2 \tag{9}$$

This is the equation of a circle of radius R in the $y - z$ plane whose center moves along the y-axis with speed $u = \omega R = E/B$.

The trajectory of the particle moves like a point on the rim of a circle whose center moves along the y-axis with speed u – the motion starting in such a way that the point is at the origin at $t = 0$. This trajectory is called a*cycloid*.

Problem 1

Show that equations 5 and 6 are the solutions to 3 and 4.

Problem 2

Show that equation 9 follows from equations 7 and 8.

Problem 3

Using *Mathematica* make a plot of the trajectory of a proton in crossed $\vec{E}$ and $\vec{B}$ fields where the magnitudes of the fields are 10M V/m and 1 T respectively. Show the first four cycles of the motion. You will need to use the `ParametricPlot` function.