A. C. Circuits - Part I

Honors Physics Note 004

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Honors Physics P222 - Spring, 2004

Introduction

In this note we treat a.c. circuits, *i.e.* circuits for which the voltage and current vary sinusoidally, like $\cos\omega t$. In order to analyze such circuits we make use of complex numbers. Circuit elements such as the capacitor and inductor have the equivalent of the resistance for resistors – the equivalent quantities being called impedances. Whereas the impedance (resistance) for a resistor is pure real and independent of frequency ω, the impedances for a capacitor and inductor is pure imaginary and dependent on frequency. Combinations of impedances yield a net impedance which is complex, that is, with a real and imaginary component. The use of complex numbers makes it easy to track the phase between the voltage and current through various parts of an a.c. circuit.

This note includes a review of complex numbers and their representation, rules for combining impedances and treatment of specific circuits like the series RLC circuit and high-pass and low-pass filters.

Representing complex numbers

Before discussing *alternating current – a.c.* circuits we review the mathematics of complex numbers. We will represent a complex number by variable with a *hat* over the complex number such as $\hat{z}$. One way to represent a complex number is as follows:

$$\hat{z} = a + ib \tag{1}$$

Where a is the *real* part and b the *imaginary* part and of course $i = \sqrt{-1}$. An alternative way to write a complex number follows by use of Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$

Before we move on let us dwell for a moment on equation 2. One way understand the formula is to use the series expansion for the exponential function:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \tag{3}$$

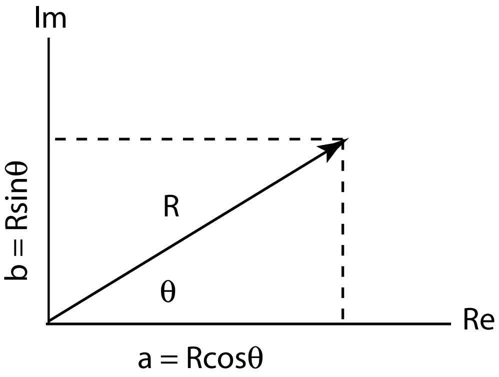

Figure 1: A complex number $\hat{z}$ can be represented as a vector in two-dimensional space where the horizontal axis is the *Real* axis and the vertical axis is the *Imaginary* axis. The algebraic representations are: $\hat{z} = a + ib = Re^{i\theta}$ where $a = R\cos\theta$ and $b = R\sin\theta$.

After expanding and grouping the terms into those containing i and the remainder, the latter is the series for $\cos\theta$ and the former, after factoring out i, is the series for $\sin\theta$. Euler's formula also leads to the following lovely result:

$$e^{i\pi} + 1 = 0 \tag{4}$$

that contains in one place the important numbers in mathematics.

An alternative way of writing $\hat{z}$ is then:

$$\hat{z} = Re^{i\theta} \tag{5}$$

where $R = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$. Figure 2 helps make the connection. A complex number $\hat{z}$ can be represented as a vector in two-dimensional space where the horizontal axis is the *Real* axis and the vertical axis is the *Imaginary* axis.

It is worthwhile to point out that an equation involving complex variables always implies two equations, one for the real part and one for the imaginary part just as an equation involving three-dimensional vectors implies three separate equations.

Complex numbers and sinusoidal oscillations

A.C. voltage sources, like the standard household voltage provided by power companies in the US, vary sinusoidally: $V(t) = V_0 \cos\omega t$, where for $f = 60\ Hz$, $\omega = 2\pi f = 377\ s^{-1}$. The amplitude V_0 is related to the *rms* voltage, V_{rms} in the following way: $V_0 = \sqrt{2}V_{rms}$. V_{rms} for household power is 110 V as measured by a voltmeter. Since the voltage varies like $cos\omega t$ one uses the square root of the average (over a complete cycle) of the square of the voltage as a measure of the voltage. The simple average of the voltage over a cycle would of course always be zero.

The average of $\cos^2 \omega t$ over a complete cycle is of course equal to the average of $\sin^2 \omega t$ and since $\sin^2 \omega t + \cos^2 \omega t = 1$ then either average is 1/2. So $V_{rms} = V_0/\sqrt{2}$.

Of course when we write $V(t) = V_0 \cos \omega t$ it should be understood that we could have also written voltage as $V(t) = V_0 \sin \omega t$ – it all depends on how we define $t = 0$. In general:

$$V(t) = A \cos \omega t + B \sin \omega t = V_0 \cos(\omega t + \delta) \tag{6}$$

Both forms are equivalent and either A and B or V_0 and δ will completely specify $V(t)$. It all depends on how $t = 0$ is defined. We refer to δ as the *phase angle*.

Using Euler's formula we can also write the voltage as $V(t) = V_0 e^{i\omega t}$. We saw in d.c. circuits that the current through and the voltage across a resistor are related simply: $V = IR$ so we expect that for a.c. voltage that:

$$V_0 e^{i\omega t} = R I_0 e^{i\omega t} \tag{7}$$

So the current also varies sinusoidally. It also happens that the current and voltage are in phase in a resistor and this happens because the resistance R is purely a real number. We will soon see that for a.c. circuits we can have elements or combination of elements for which the voltage and phase will no longer be in phase.

Capacitors and a.c. voltage

For a capacitor the voltage and charge are related by $Q = CV$. For a sinusoidally varying current, $I(t) = I_0 e^{i\omega t} = dQ/dt$, we integrate the current to obtain:

$$Q = \int I_0 e^{i\omega t} dt = \frac{I_0 e^{i\omega t}}{i\omega} = C V_0 e^{i\omega t} \tag{8}$$

From the above we see that $V_0 e^{i\omega t} = I_0 e^{i\omega t}/(i\omega C))$ where $1/i\omega C$ is a quantity that obviously has units of resistance. We call this quantity an *impedance* and denote it as $\hat{Z}_C$ so for a capacitor the relation between voltage and current is linear with the proportionality constant being:

$$\hat{Z}_C = \frac{1}{i\omega C} \tag{9}$$

Unlike resistance, the capacitive impedance is pure imaginary and it depends on frequency. In the limit that $\omega \rightarrow 0$, d.c. voltage, the impedance is infinite – no current flows when a constant voltage exists across a capacitor.

The implication of a purely imaginary impedance is that the voltage and current are out of phase by $\pi/2$ since $1/i = e^{-i\pi/2}$. The voltage and current in a capacitor can be written as:

$$V_0 e^{i\omega t} = \frac{I_0}{\omega C} e^{i(\omega t - \pi/2)} \tag{10}$$

Inductors and a.c. voltage

For and inductor, the voltage V, time rate of change of current dI/dt and inductance L are related by:

$$V = L\frac{dI}{dt} \tag{11}$$

For $I(t) = I_0 e^{i\omega t}$ we find that $dI/dt = i\omega I$ so the voltage across the inductor is given by $V_0 e^{i\omega t} = i\omega L I_0 e^{i\omega t}$ implying that the impedance for the inductor is:

$$\hat{Z}_L = i\omega L \tag{12}$$

Like for the capacitor, the impedance of an inductor is pure imaginary and the voltage and current are out of phase by $\pi/2$:

$$V_0 e^{i\omega t} = \omega L I_0 e^{i(\omega t + \pi/2)} \tag{13}$$

Impedances in general

Figure 2 summarizes the expressions for the impedances of a resistor, inductor and capacitor.

There is beauty in the simplicity with which one handles impedances in a circuit. The rules that apply for treating for resistors. For two impedances in series:

$$\hat{Z}_{eq} = \hat{Z}_1 + \hat{Z}_2 \tag{14}$$

and for two impedances in parallel:

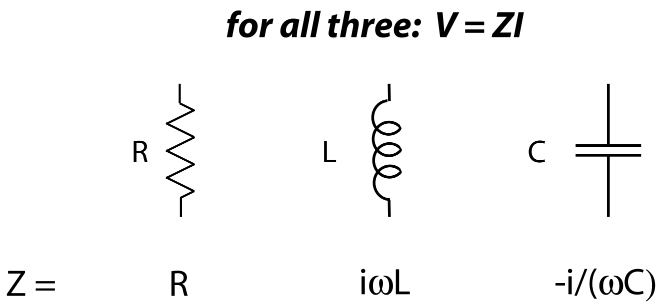

Figure 2: Impedances for a resistor, inductor and capacitor

$$\hat{Z}_{eq} = \frac{\hat{Z}_1 \hat{Z}_2}{\hat{Z}_1 + \hat{Z}_2} \tag{15}$$

RLC circuit

As an example, consider the circuit of Figure 3 where we have a series combination of a resistor, capacitor and inductor across an a.c. voltage source and the equivalent circuit. Using the rules for combining impedances the equivalent impedance of R - C - L in series is $\hat{Z}_{eq}$ where:

$$\hat{Z}_{eq} = R + i\left(\omega L - \frac{1}{\omega C}\right) \tag{16}$$

The equivalent impedance can also be written in the form:

$$\hat{Z}_{eq} = Z_0 e^{i\delta} \tag{17}$$

where:

$$Z_0 = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{18}$$

and

$$\delta = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) \tag{19}$$

So if the voltage source is given by $V_0 e^{i\omega t}$ then the current flowing through the source is $(V_0/Z_0)e^{(i\omega t - \delta)}$

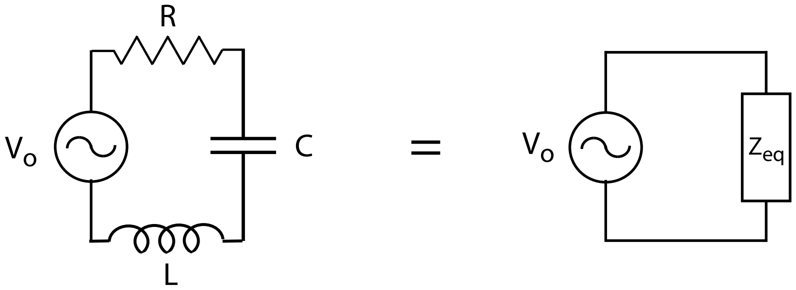

Figure 3: A circuit with a resistor, capacitor and inductor in series across an a.c. voltage source and its equivalent circuit.

– in other words, the current through the power source and the voltage across the power source are not in phase. We will explore the implications for the power delivered later in this note.

There are several interesting features of this circuit that follow from our analysis that we enumerate below. For simplicity we will label the currents and voltages through the power source and the other circuit elements by the subscripts: S, R, C and L.

1. The currents $\hat{I}_S$,$\hat{I}_R$,$\hat{I}_C$ and $\hat{I}_L$ are all in phase with each other.
2. The voltages $\hat{V}_S$,$\hat{V}_R$,$\hat{V}_C$ and $\hat{V}_L$ are all out of phase with each other.
3. The sum of $\hat{V}_R$,$\hat{V}_C$ and $\hat{V}_L$ is in phase with $\hat{V}_S$.
4. $\hat{V}_R$ is in phase with $\hat{I}_R$.
5. $\hat{V}_C$ is 90^o out of phase with $\hat{I}_C$.
6. $\hat{V}_L$ is -90^o out of phase with $\hat{I}_L$.
7. $\hat{V}_L$ is 180^o out of phase with $\hat{V}_C$.

An examination of equations 18 and 19 shows that for the special case:

$$\omega = \omega_0 = \sqrt{\frac{1}{LC}} \tag{20}$$

$Z_0 = R$ and $\delta = 0$ – as if the inductor and capacitor were replaced by wires. We refer to ω_0 are the *resonant frequency.* The equivalent impedance is pure real – but only at the resonant frequency. In that case $\hat{V}_S$ is in phase with $\hat{I}_S$ and $\hat{V}_S$ is in phase with $\hat{V}_R$.

Another arrangement of a resistor, capacitor and inductor is shown in Figure 4. In one of the problems you will be asked to compute the equivalent impedance and to examine other properties of this circuit.

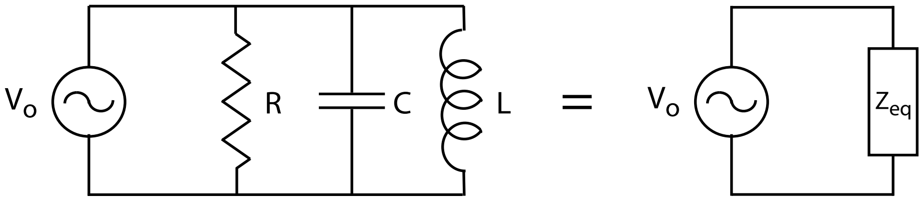

Figure 4: A circuit with a resistor, capacitor and inductor all in parallel across an a.c. voltage source and its equivalent circuit.

Power

For d.c. voltage and current the power dissipated by a resistor R is the voltage across the resistor times the current through the resistor or VI. For a.c. voltage and current the power dissipated by an impedance is the time average of the product of the voltage times the current. Similarly, the power supplied by an a.c. voltage source will also be the time average of the product of the voltage times the current.

Let us consider a specific example. Suppose the voltage is given by $V(t) = V_0 \cos(\omega t)$ and the current by $I(t) = I_0 \cos(\omega t + \delta)$. The amplitudes for current and voltage are related to the *r.m.s.* values by $I_0 = \sqrt{2} I_{rms}$ and $V_0 = \sqrt{2} V_{rms}$. We are interested in the time average, over a complete cycle, of the power $P_{av}(t)$ where:

$$P_{av}(t) = V_0 I_0 \langle cos(\omega t) \cos(\omega t + \delta) \rangle \tag{21}$$

Equivalently:

$$P_{av}(t) = V_{rms} I_{rms} \cos\delta \tag{22}$$

The steps leading from equation 21 to 22 are left as a problem. You will need to use the result that the average of a complete cycle of $\cos^2 \omega t$ is 1/2. The same result, equation 22 , would have been obtained had we started with $V(t) = V_0 \sin(\omega t)$ and $I(t) = I_0 \sin(\omega t + \delta)$.

Recall that δ is determined by the net equivalent impedance appearing across the voltage source. If we consider the current through and the voltage across individual components we note that for a capacitor or inductor $\delta = \pm\pi/2$ so that the average power dissipation in these elements is zero. For a resistor $\delta = 0$ and the power dissipation is then just $I_{rms}^2 R$. So if the impedance is pure imaginary there is no net power dissipation.

Low-pass and high-pass filters

A simple combination of a resistor and capacitor can be used to build a low-pass or high-pass filter as shown in Figure 5. The ratio of the output to input voltages depends on frequency. In the limit

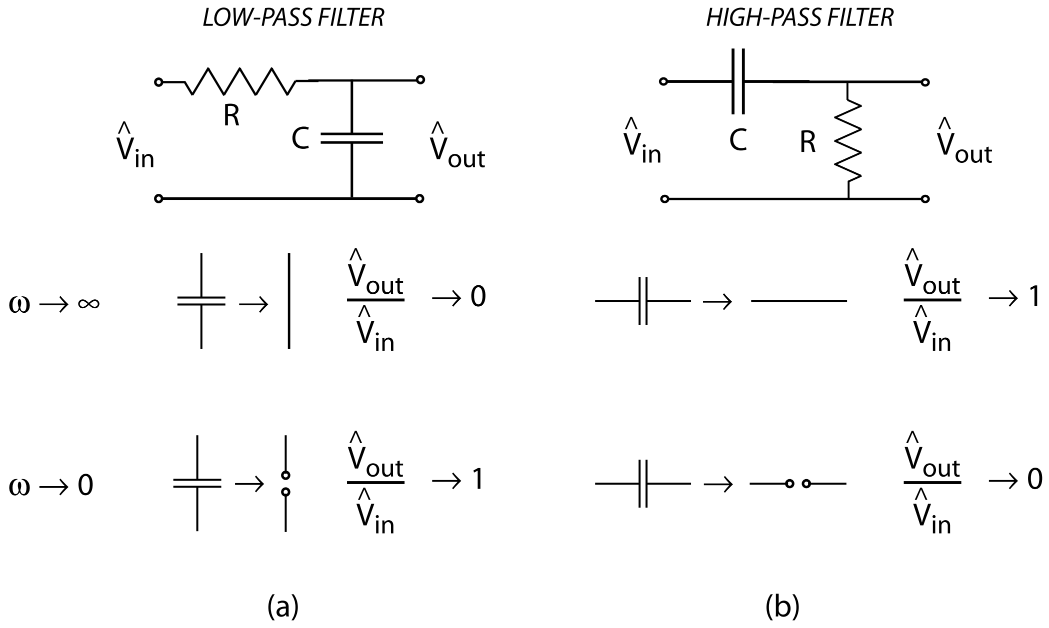

Figure 5: Two arrangements of a resistor and capacitor to form a (a) low-pass and (b) high-pass filter. The ratio of the output to input voltages depends on frequency. In the limit $\omega \rightarrow \infty$ the capacitor behaves like a wire or short circuit. In the limit that $\omega \rightarrow 0$ the capacitor behaves like a broken wire or open circuit. Thus the low-pass filter passes low-frequency voltages and blocks high frequency voltages while the high-pass filter does just the opposite.

$\omega \rightarrow \infty$ the capacitor behaves like a wire or short circuit because $|\hat{Z}_C| = 1/\omega C \rightarrow 0$. In the limit that $\omega \rightarrow 0$ the capacitor behaves like a broken wire or open circuit because $|\hat{Z}_C| = 1/\omega C \rightarrow \infty$. Thus the low-pass filter passes low-frequency voltages and blocks high frequency voltages while the high-pass filter does just the opposite.

These are the limiting cases. For intermediate frequencies it is useful to look at either circuit of Figure 5 as a voltage divider. As a reminder we show a voltage divider made of resistors in Figure 6(a). The input voltage appears across the series combination of R_1 and R_2 while the output voltage is that across R_2. The ratio of output to input voltages is given by:

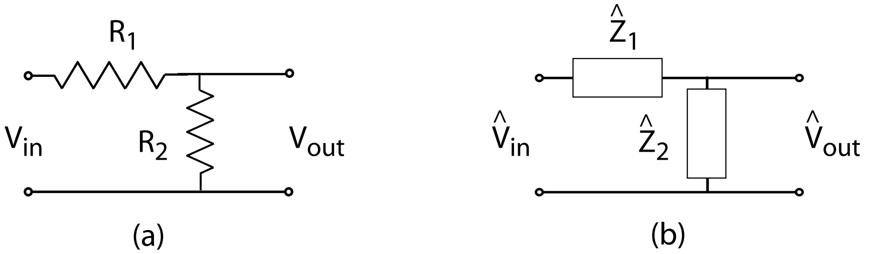

Figure 6: (a) A resistive voltage divider circuit. The input voltage appears across the series combination of R_1 and R_2 while the output voltage is that across R_2. (b) A voltage divider circuit in which the resistances R_1 and R_2 are replaced by $\hat{Z}_1$ and $\hat{Z}_2$ respectively.

$$\frac{V_{out}}{V_{in}} = \frac{R_2}{R_1 + R_2} \tag{23}$$

In Figure 6(b) the resistances R_1 and R_2 are replaced by $\hat{Z}_1$ and $\hat{Z}_2$ respectively and the ratio of output to input voltages is given by:

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{\hat{Z}_2}{\hat{Z}_1 + \hat{Z}_2} \tag{24}$$

Starting with the low-pass filter of Figure 5(a) equation 24 becomes:

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{1/i\omega C}{R + 1/i\omega C} = \frac{1}{1 + i\omega RC} \tag{25}$$

But because we can write $1 + i\omega RC$ as $\sqrt{1 + (\omega RC)^2} \cdot e^{i\theta}$ where $\theta = \tan^{-1}(\omega RC)$ then:

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{1}{\sqrt{1 + (\omega RC)^2}} e^{-i\theta} \tag{26}$$

Note that θ is the phase angle between the output and input voltages. As expected, equation 26 shows that the ratio of output to input voltage vanishes and $\theta \rightarrow \pi/2$ as $\omega \rightarrow \infty$ while the ratio approaches one and $\theta \rightarrow 0$ as $\omega \rightarrow 0$.

Note that when $\omega = 1/RC$ the ratio of the magnitude of the output voltage to input voltage is $1/\sqrt{2}$ and the phase angle between the two is $\pi/4$.

It will be left as an exercise for you to show that for the high-pass filter of Figure 5(b) equation 24 yields:

$$\frac{\hat{V}_{out}}{\hat{V}_{in}} = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} e^{-i\theta} \tag{27}$$

where now $\theta = \tan^{-1}(-1/\omega RC)$ and that this yields the correct limiting behavior as $\omega \rightarrow \infty$ and $\omega \rightarrow 0$.

Figures 7 and 8 show the dependence on ωRC of the ratio of the magnitude of the output voltage to input voltage and the phase angle between the two respectively.

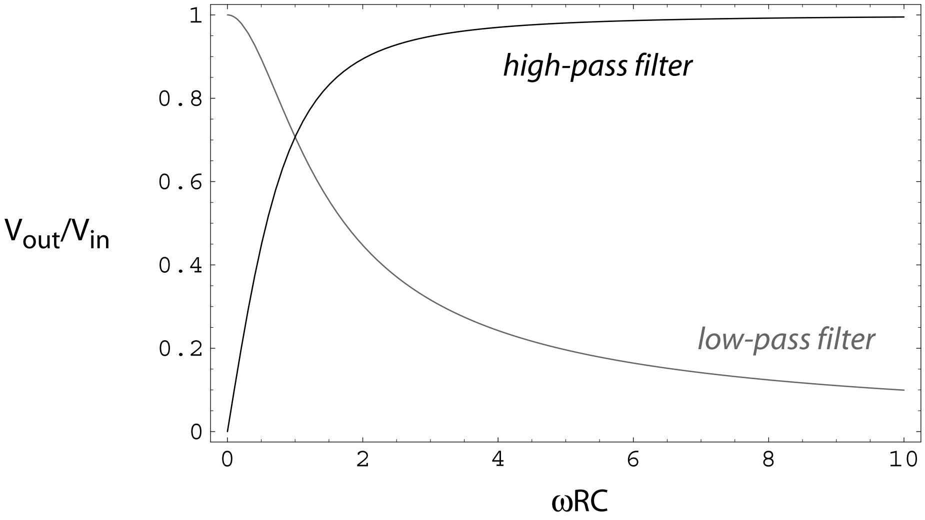

Figure 7: Dependence of the ratio V_{out}/V_{in} as a function of ωRC for a high-pass filter (dark curve) and a low-pass filter (gray curve).

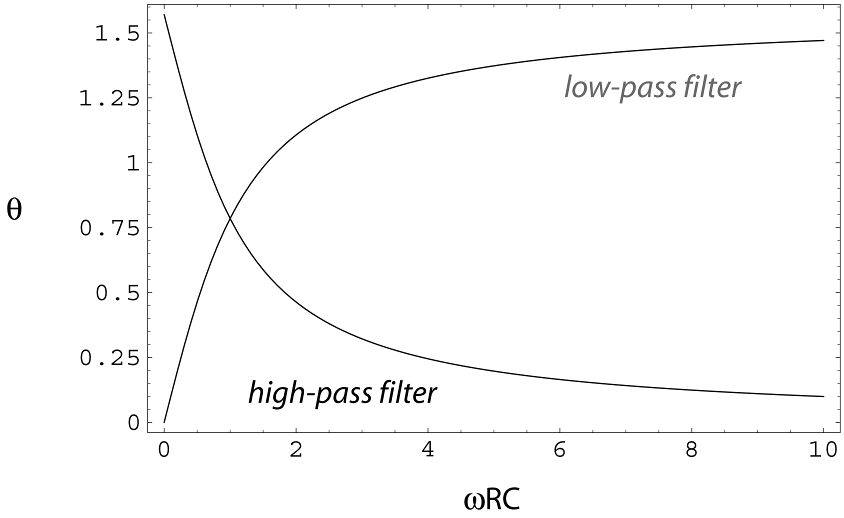

Figure 8: Dependence of the phase angle θ between the output voltage V_{out} and input voltage V_{in} as a function of ωRC for a high-pass filter (dark curve) and a low-pass filter (gray curve).

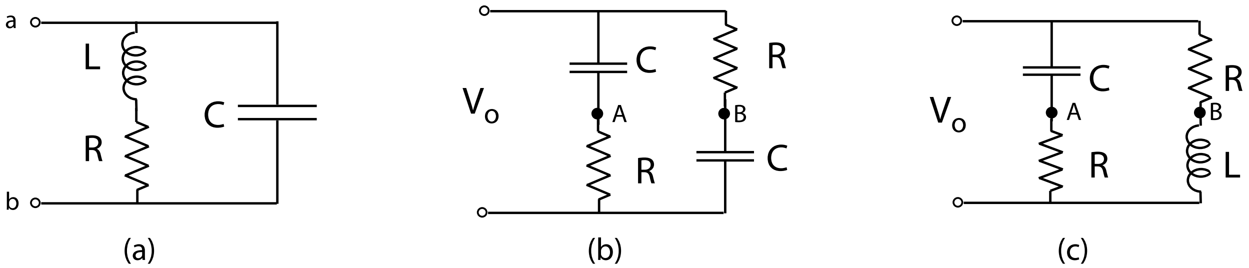

Figure 9: Figures for (a) Problem 3; (b) Problem 4; and (c) Problem 5

Problems

Problem 1

Use Euler's formula, equation 2, to find an expression for $\sin 4x$ in terms of $\sin x$ and or $\cos x$.

Problem 2

(a) Show that for the high-pass filter of Figure 5(b) equation 24 yields equation 27. (b) When $\omega = 1/RC$, what is the ratio of the magnitude of the output voltage to input voltage? (c) When $\omega = 1/RC$, what is the phase angle between the output voltage and input voltage?

Problem 3

FIgure 9(a) shows a circuit consisting of an series combination of an inductor and capacitor in parallel with a capacitor. Is is possible to find a condition that would result in an impedance between terminals a and b being completely real? If the answer is yes, what is that condition?

Problem 4

In FIgure 9(b) the input voltage is $\hat{V}_0$ and the voltage between points A and B is $\hat{V}_{AB}$. Show that for any frequency $|\hat{V}_0|^2 = |\hat{V}_{AB}|^2$. Also, find the frequency for which $\hat{V}_0$ and $\hat{V}_{AB}$ are out of phase by $\pi/2$.

Problem 5

What is the relationship among R, L and C that will guarantee for the circuit of Figure 9(c) that the voltage difference between points A and B, $\hat{V}_{AB}$, is zero for any frequency.

Problem 6

Consider the circuit of Figure 4. (a) Find an expression for the impedance in the form $Z_0 e^{i\delta}$ specifying Z_0 and δ. (b) For what, if any, frequency is the impedance pure real, and in that case is the magnitude of impedance a minimum or maximum? (c) What phase relationship among the voltages across the power source and the individual circuit elements? (d) Same as part (c) but this time for the currents.

Problem 7

FIll in the steps leading from equation 21 to 22. Show that the same result, equation 22 , would have been obtained had we started with $V(t) = V_0 \sin(\omega t)$ and $I(t) = I_0 \sin(\omega t + \delta)$.

Note that Problems 3, 4 and 5 are very similar to problems found in Chapter 8 in *Electricity and Magnetism* by E. M. Purcell – 2nd edition. This is volume 2 of the Berkely Physics Course.