RC Circuits

Honors Physics Note 002

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Honors Physics P222 - Spring, 2004

Introduction

This note concerns the behavior of circuits that include combinations of resistors, capacitors and possibly a battery.

Capacitors

Consider two parallel conducting plates of area A and separation d as shown in Figure 1 (a). Such an arrangement is called a *capacitor*. If a battery of voltage V is connected across this arrangement of conductors charge $+Q$ will collect on one plate and equal and opposite charge $-Q$ on the other plate. If the voltage is V the magnitude of charge is

$$Q = CV \tag{1}$$

where C is the capacitance of the capacitor. The capacitance depends only on the geometry of the plates (area and separation) and on the nature of the material between the plates. In SI units where we use coulombs for charge and meters for distance the formula for the capacitance of a parallel plate capacitance is given by:

$$C = \frac{\epsilon_o A}{d} \tag{2}$$

Here we assume that the space between the plates is filled with vacuum (air is a good enough approximation). The constant ϵ_o, called the *permittivity* of free space, determines the scale of the force between two charges separated by a fixed distance and has the value $8.85 \times 10^{-12}\ C^2 N^{-1} m^{-2}$. The unit of capacitance is the *farad* determined by F. It turns out that the *farad* is unwieldy – for example if two circular plates are separated by a millimeter, the radius of the plates needed for a 1 F capacitor would be 6 km! Typical capacitors are measured in units of microfarads ($\mu F = 10^{-6}\ F$) or picofarads ($pF = 10^{-12}\ F$). The symbol of a capacitor is shown in Figure 1 (b).

In this course we will calculate the capacitance for various geometries (*e.g.* parallel plates, coaxial cylinders, parallel wires, etc..). One technique is integrate the energy density stored in the electric field over the volume of the capacitor. The energy density is given by $\epsilon_o E^2/2$ where E is the electric field and is proportional to charge Q.

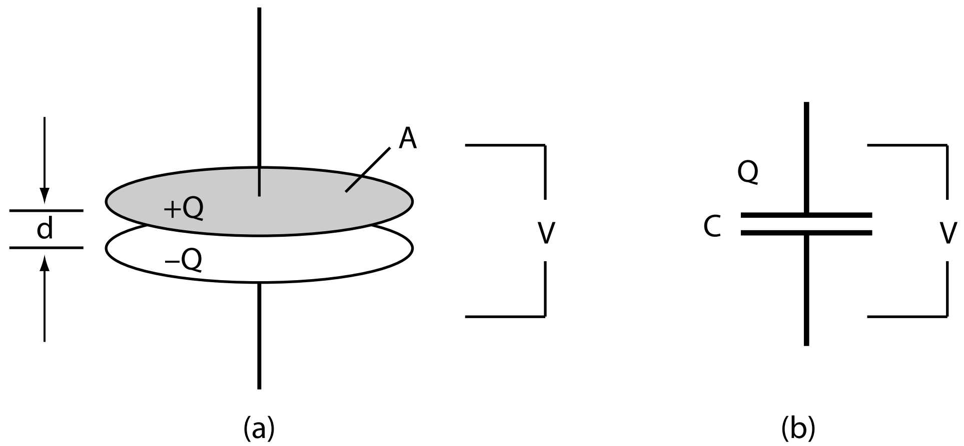

Figure 1: (a): Schematic of a parallel plate capacitor – the separation of the plates is d and the area is A. (b) Symbol for a capacitor. The voltage across the capacitor is V, the capacitance is C and charge Q and $Q = CV$.

Series combination of capacitors

The left circuit of Figure 2 shows two capacitors C_1 and C_2 connected in series and the right circuit shows the equivalent circuit with the series combination replaced by its equivalent capacitor C_{eq}. The voltage difference between the terminals a and b is V.

Because of the conservation of charge the charge on C_1 is the same as the charge on C_2 and the same as the charge on C_{eq} and equal fo Q. The voltage on the each of the two capacitors in series add to give the total voltage V. But $V_1 = Q/C_1$, $V_2 = Q/C_2$ and $V = Q/C_{eq}$ which leads to:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{3}$$

or

$$C_{eq} = \frac{C_1 C_2}{C_1 + C_2} \tag{4}$$

Note that the rule for finding the equivalent of series capacitors is similar to the rule for finding the equivalent of parallel resistors.

Parallel combination of capacitors

The left circuit of Figure 3 shows two capacitors C_1 and C_2 connected in series and the right circuit shows the equivalent circuit with the parallel combination replaced by its equivalent capacitor C_{eq}. The voltage difference between the terminals a and b is V.

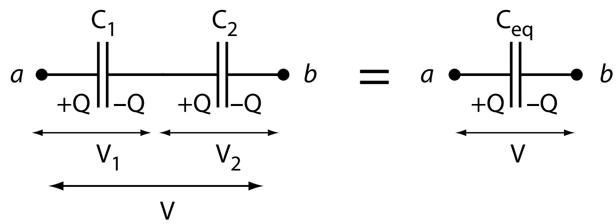

Figure 2: The left circuit shows two capacitors C_1 and C_2 connected in series and the right circuit shows the equivalent circuit with the series combination replaced by its equivalent capacitor C_{eq}. The voltage difference between the terminals a and b is V.

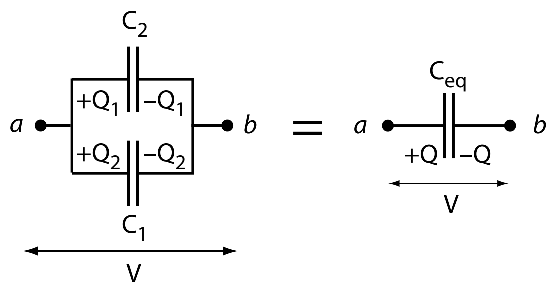

Figure 3: The left circuit shows two capacitors C_1 and C_2 connected in parallel and the right circuit shows the equivalent circuit with the parallel combination replaced by its equivalent capacitor C_{eq}. The voltage difference between the terminals a and b is V.

In this case the voltage across C_1 is the same as that across C_2 and the same as that across C_{eq}. But the charge Q_1 on C_1 and the charge Q_2 on C_2 add to yield the charge Q on C_{eq}. This leads to:

$$C_{eq} = C_1 + C_2 \tag{5}$$

Note that the rule for finding the equivalent of parallel capacitors is similar to the rule for finding the equivalent of series resistors.

Discharging capacitor

In the circuit of Figure 4 shows a capacitor C is connected across a resistor R using a switch S. Initially the capacitor carries charge Q_0. At $t = 0$ the switch is closed and the capacitor discharges through the resistor. The charge on the capacitor, $Q(t)$, now varies with time as does the current, $i(t)$, through the resistor. Positive charge flows from the positive plate of the capacitor to the negative plate until the net charge on each plate vanishes. The direction of the current flow is also consistent with the voltage charge across the resistor being the same as the voltage change across the capacitor:

$$\frac{Q(t)}{C} = i(t)R \tag{6}$$

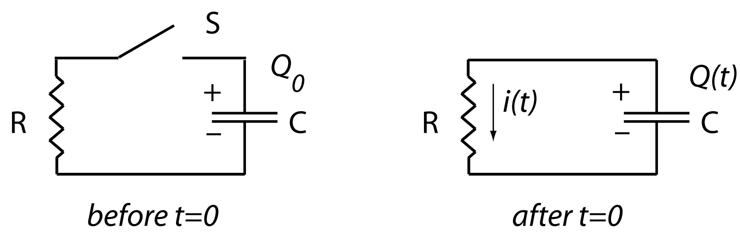

Figure 4: A capacitor C is connected across a resistor R using a switch S. Initially the capacitor carries charge Q_0. At $t = 0$ the switch is closed and the capacitor discharges through the resistor. The charge on the capacitor, $Q(t)$, now varies with time as does the current, $i(t)$, through the resistor.

Since $i = -dQ/dt$ – the charge is decreasing explaining the minus sign – leading to:

$$\frac{dQ}{Q} = -\frac{dt}{RC} \tag{7}$$

Integration:

$$\int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt \tag{8}$$

yields:

$$\ln\left(\frac{Q}{Q_0}\right) = -\frac{t}{RC} \tag{9}$$

or:

$$Q(t) = Q_o e^{-t/RC} \tag{10}$$

where Q_o is the initial charge on the capacitor. The charge decreases exponentially. The current also decreases exponentially:

$$i(t) = -\frac{dQ}{dt} = \frac{Q_o}{RC}e^{-t/RC} = \frac{V_o}{R}e^{-t/RC} = i_o e^{-t/RC} \tag{11}$$

Note that the product of R times C must have units of s. The so-called RC *time constant* sets the time scale for a capacitor to discharge or – as we will see – to discharge. The time required for the capacitor to lose one-half its charge is $t_{1/2} = \ln(2)RC = 0.693RC$.

Energy stored in a capacitor

Clearly as the capacitor discharges current flows though the resistor until the capacitor is fully discharged and the current goes to zero. But current flowing through resistor leads to a power dissipation, $P(t)$, that varies with time:

$$P(t) = i^2(t)R \tag{12}$$

Integrating the power from $t = 0$ to $t = \infty$ yields the total initial energy stored in the capacitor:

$$\int_0^\infty P(t)dt = \frac{Q_o^2}{R^2C^2}\int_0^\infty e^{-2t/RC} R dt = \frac{Q_o^2}{2C} \tag{13}$$

So a capacitor C with charge Q and voltage V has stored energy that can be expressed as:

$$E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \tag{14}$$

where we have made use of $Q = CV$.

A capacitor, resistor and battery in series

In Figure 5 a capacitor C is connected in series with a resistor R across a battery V using a switch S. Initially the capacitor carries zero charge Q. At $t = 0$ the switch is closed and the capacitor charges through the resistor. The charge on the capacitor, $Q(t)$, now varies with time as does the current, $i(t)$, through the resistor. At $t = 0$ the charge on the capacitor is zero and so is the voltage difference across the capacitor so when the switch is closed the voltage V provided by the battery appears across the resistor so the initial current through the resistor is V/R. As the capacitor starts to charge up a voltage appears across the capacitor and the net voltage across the resistor is the battery voltage minus the capacitor voltage which means that the current through the resistor decreases with time. Eventually the capacitor is fully charged so that the voltage across the capacitor is the same as the battery voltage or the final charge is CV. At this point the voltage across the resistor is zero so no more current flows.

To find the time-dependence of the charge $Q(t)$ and current $i(t)$ we start with the expression for the voltage drops in making a complete loop of the circuit on the right hand side of Figure 5:

$$V - \frac{Q}{C} - iR = 0 \tag{15}$$

Recalling that $i = dQ/dt$ you can verify that the solution is given by:

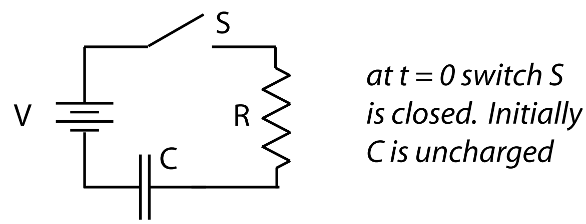

Figure 5: A capacitor C is connected in series with a resistor R across a battery V using a switch S. Initially the capacitor carries zero charge Q. At $t = 0$ the switch is closed and the capacitor charges through the resistor. The charge on the capacitor, $Q(t)$, now varies with time as does the current, $i(t)$, through the resistor.

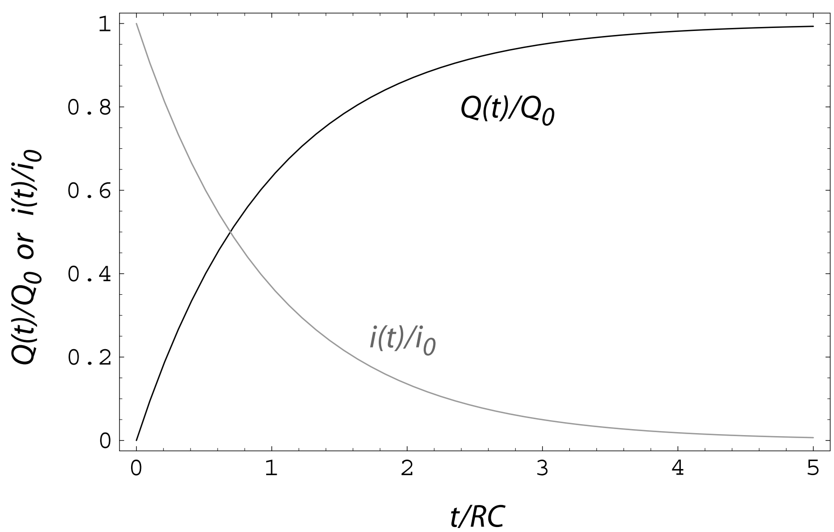

Figure 6: Plot of $Q(t)/Q_f$ [black curve - see eq. 16] and $i(t)/i_0$ [gray curve - see eq. 17] as a function of t/RC for the charging capacitor circuit of Figure 5.

$$Q(t) = CV(1 - e^{-t/RC}) = Q_f(1 - e^{-t/RC}) \tag{16}$$

substitution of eq. 16 into eq. 15. Indeed we see that $Q(t = 0) = 0$ and $Q(t = \infty) = CV = Q_f$. The current is obtained by differentiating eq. 16:

$$i(t) = \frac{V}{R}e^{-t/RC} = i_0 e^{-t/RC} \tag{17}$$

from which we see that the initial current is $V/R = i_0$ as expected and eventually the current goes to zero.

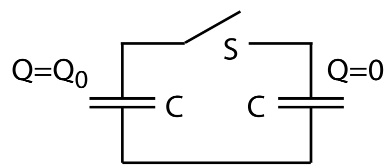

Figure 7: Problem 1

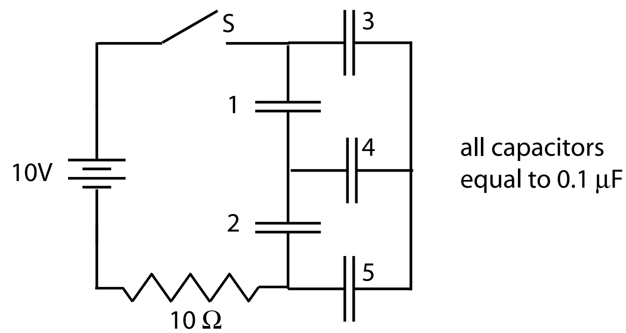

Figure 8: Problem 2

Problems

Problem 1

Fiigure 7 shows two identical capacitors of capacitance C. One of the capacitors carries charge $Q = Q_0$ and $Q = 0$. Initially the switch is open and then closed. (a) How much charge exists on each capacitor after the switch has been closed for some time. (b) What is the total energy before and after the switch is closed and account for any difference.

Problem 2

Figure 8 shows five capacitors, all identical and with capacitance equal to 0.10 μF. The network of capacitors is connected across a series combination of a 10 V battery and a 10 Ω resistor when switch S is closed. Initially all the capacitors are uncharged. A $t = 0$ the switch is closed and sufficient time all capacitors are fully charged. (a) What is the final charge on each capacitor? (b) While the capacitors were being charged what was the total energy supplied by the battery? (c) While the capacitors were being charged what was the total energy dissipated in the resistor? (d) What is the final total stored energy in the capacitors?