D. C. Circuits

Honor Physics Note 001

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Honors Physics P222 - Spring, 2004

Introduction

This is the first in a series of short notes on circuit theory. Normally *direct current* or *d.c.* circuits and *alternating current* or *a.c.* circuits are discussed midway during the usual introductory course on electricity and magnetism. But I will break from tradition and discuss circuits first since we will be using electrical circuits in the early labs associated with this course. You will be making quantitative measurements and comparing these with theoretical expectations. Actually I think starting off this course with circuit theory makes some sense because you really do not need to discuss the concepts of the *electric field* and *magnetic field* that normally precede circuits in order to understand how circuits operate – at least at the practical and operational level. And most of us have some notion of electrical *current* and *voltage* and some familiarity with devices such as batteries, resistors, capacitors and inductors and the ubiquitous sources of a.c. power all around us. After we discuss the electric and magnetic fields we will return to these circuits and their elements to understand them at a more fundamental level.

Electrical charge, current and voltage

We will start our discussion of electrical circuits by considering simple circuits in which some combinations of resistors are connected by wires across a battery. The battery supplies a certain voltage and currents will flow through the battery, the resistors and the wires connecting everything together. Later we will also include capacitors that can also store electric charge. The relevant physical variables we will dealing with are *charge*, *current* and *voltage*.

Electric charge is measured in *coulombs* denoted by C and is always an integral multiple of $|e|$, the charge on the electron where $|e| = 1.6 \times 10^{-19}\ C$. Remarkably, the fundamental charge of many of the so-called elementary particles is either negative or positive but with the same magnitude of charge $|e|$, even though their substructures are completely different.

Current measures charge flowing through some cross-sectional area per unit time and is measured in *amperes* denoted by A where $1\ A = 1\ C/s$. In the metal conductors that form the connecting wires making up circuits, the charge carriers are electrons and the number of electrons passing through a point in the wire is large. For example, if a wire carries $1\ A$ of current, approximately 6×10^{18} electrons pass through any point every second.

Voltage is specified in units of *volts* denoted by V and is a measure of energy per unit charge delivered by some device called a *power supply* or a *voltage source* or a *battery.* For now we will deal with *d.c.* voltage sources that supply a voltage constant in time.

After we cover electric and magnetic fields we will see a nice parallelism between the electric field and voltage on one hand and the magnetic field and current on the other hand.

A simple circuit

Figure 1 shows a simple circuit consisting of a voltage source supplying voltage V to a resistor R. The battery is connected to the resistor with wires that are assumed to have zero resistance. The arrows indicate the direction of current I.

The element in the left leg of the circuit is the symbol for a battery or voltage source. Note the convention for the positive and negative terminals of the battery. The element in the right leg is the symbol for a resistor.

The direction of the current arrow indicates the direction of the flow of positive charges from the positive terminal of the battery to the negative terminal. Negative charges flow in the opposite direction for the same direction of the current arrow.

Points a and b are at the same voltage since there is no voltage drop across an ideal wire. Indicating the voltage at a by V_a and at b by V_b we have $V_a = V_b$. Similarly, $V_c = V_d$. We also have $V_a - V_c = V = V_b - V_d$. Notice that the as we follow the current across the resistor we go from a higher voltage V_b to a lower voltage V_d. *Moving across a resistor along the current results in a voltage drop.*

The relation between the voltage V_R across a resistor and the current I_R through the resistor is given by Ohm's Law:

$$V_R = I_R \cdot R \tag{1}$$

The unit of resistance is the ohm, denoted by Ω. If 1 A of current flows through a 1 Ω resistor the voltage drop is 1 V.

There are two subtle points about the voltage and current in the simple circuit of Figure 1. The current that flows through the wire from point a to b is the same as the current through the resistor and is the same as the current that flows through the wire from point d to c and the same as the current that flows through the battery. This is due to the conservation of charge – at any point the charge flowing in per time is equal to the charge flowing out per time.

Regarding the voltage, starting at point a and moving to point b the voltage change is zero. Moving across the resistor from point b to d the voltage change is $-V$ – a voltage drop. Moving from point d to c there is no voltage change. Moving from point c to a across the battery the voltage change is $+V$. So the sum of all voltage changes as we make a complete closed loop around the circuit is zero. This is equivalent to the conservation of energy.

While on the subject of energy conservation it is interesting to note that as the current moves across the battery from the lower voltage terminal to the higher voltage terminal the battery does

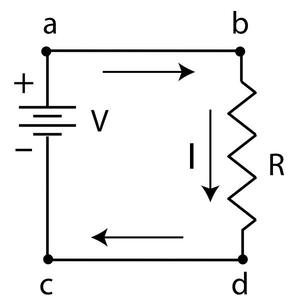

Figure 1: A simple circuit showing a voltage source supplying voltage V to a resistor R. The arrows indicate the flow of current I.

positive work on the charges and time rate of this work is power given by:

$$P = IV \tag{2}$$

Recall that voltage measures energy per charge so when multiplied by current, or charge per time the result is energy per time or power. The battery supplies 1 W of power if the current is 1 A and the voltage is 1 V. As the current passes through the resistor the material of the resistor heats up and the power dissipated in the resistor is given by the current through the resistor I times the voltage drop across the resistor $V = IR$ or:

$$P = I^2R \tag{3}$$

Resistors in series and in parallel

Figure 2 shows two resistors, R_1 and R_2, connected in *series* across a battery. In this situation the current flowing through R_1 is the same as that through R_2. The voltage across R_1 is $V_1 = V_b - V_e$ and that across R_2 is $V_2 = V_e - V_d$ and $V = V_1 + V_2$. We can replace the series combination of the two resistors with a single equivalent resistor R_{eq}, as indicated in Figure 2, and $V = V_1 + V_2$ implies that $IR_{eq} = IR_1 + IR_2$. This leads to the following rule for the equivalent of resistors in series:

$$R_{eq} = R_1 + R_2 \tag{4}$$

For n resistors in series:

$$R_{eq} = R_1 + R_2 + \ldots + R_n \tag{5}$$

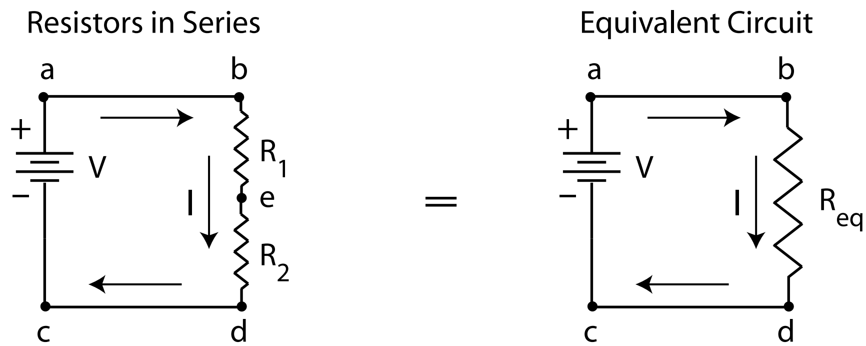

Figure 2: A circuit with a series combination of resistors and its equivalent circuit

Figure 3 shows two resistors, R_1 and R_2, connected in *parallel* across a battery. Again we want to find the equivalent equivalent resistor for the parallel combination. In this case the voltage across R_1 is the same as that across R_2 but the currents through the two resistor are not the same unless the two resistors have the same resistance. At point b the current I flowing through the battery splits into the current I_1 flowing through R_1 and the current I_2 flowing through R_2 and the conservation of charge requires that $I = I_1 + I_2$ implying that $V/R_{eq} = V/R_1 + V/R_2$ or:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{6}$$

or:

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \tag{7}$$

It is obvious that $R_{eq} < R_1$ and $R_{eq} < R_2$ and if $R_1 = R_2 = R$ then $R_{eq} = R/2$. For n resistors in parallel:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \tag{8}$$

Just to emphasize, when resistors are combined in series across a battery the current that flows through the battery is the same as the current that flows through each resistor but the voltage across each is generally different and the voltage sum is equal to the voltage across the battery. In contrast, when resistors are combined in parallel across a battery the voltage across the battery is the same as the voltage across each resistor but the current flowing through each resistor is generally different and the current sum is equal to the current flowing through the battery.

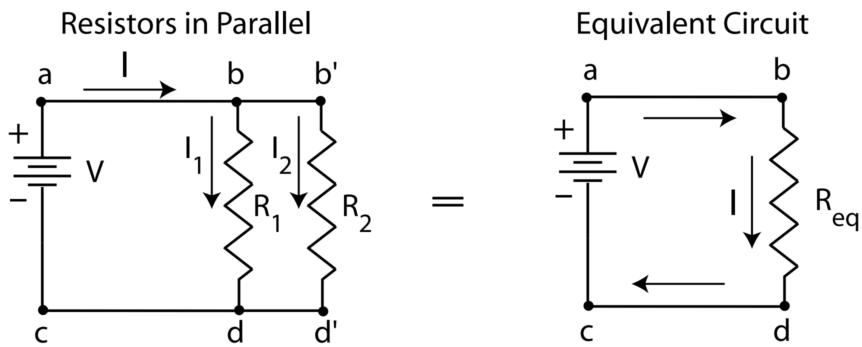

Figure 3: A circuit with a parallel combination of resistors and its equivalent circuit

Voltage and current dividers

The left circuit of Figure 2 can be viewed as a *voltage divider*. The voltage V across the battery is divided into V_1 (across R_1) plus V_2 (across R_2) where:

$$V_1 = \frac{R_1}{R_1 + R_2} V \tag{9}$$

and

$$V_2 = \frac{R_2}{R_1 + R_2} V \tag{10}$$

In contrast the left circuit of Figure 3 can be viewed as a *current divider*. The current I through the battery is divided into I_1 (across R_1) plus I_2 (across R_2) where:

$$I_1 = \frac{R_2}{R_1 + R_2} I \tag{11}$$

and

$$I_2 = \frac{R_1}{R_1 + R_2} I \tag{12}$$

Equivalent circuits

One can speak of the equivalent resistance of a network of resistors without reference to any specific voltage source that may be applied across this network. Figure 4 illustrates the application of the rule for combining series resistors and the rule for combining parallel resistors for a combination

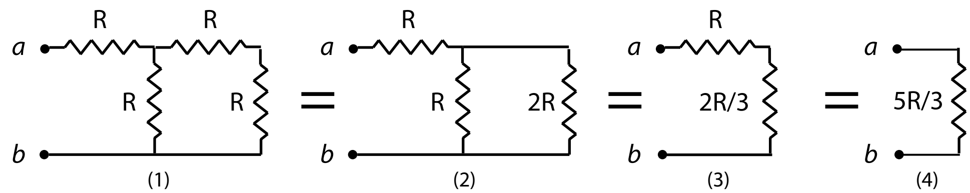

Figure 4: A combination of four resistors. To find the net resistance between the terminals a and b the rules for combining series and parallel resistors are applied.

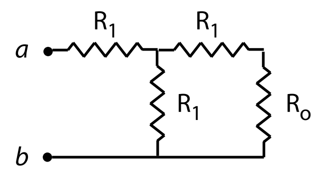

Figure 5: A combination of four resistors. The goal is to find the value of R_1 that will result in the net resistance between terminals a and b equal to R_0.

of four identical resistors to find the equivalent resistor between terminals a and b. In Figure 5 we start with a similar arrangement of four resisters, three of which are identical and equal to R_1 and the other has the value R_0. In one of the problems you are asked to find the relation between R_1 and $R0$ such that the net resistance between terminals a and b is R_0.

Figure 6 shows two combination of three resistors and for both the combinations there are three terminals: a, b and c. The values of the resistors are chosen so that the net resistance between any two terminals for one combination is equal to that between the corresponding terminals of the other combination.

And finally we consider the interesting network shown in Figure 7 and called a ladder network. The top circuit shows a repeating pattern of sections, each section including two resistors: R_1 and R_2. The goal is to find the net resistance between the terminals a and b. The key to solving this is to assume that the resistance of the network is R_0 and then to assume that adding another section (as shown in the bottom circuit of Figure 7) does not change the resistance. This is explored in one of the problems.

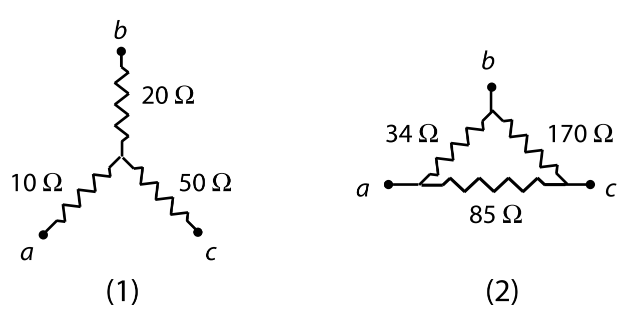

Figure 6: Two combination of three resistors. For the two combinations the net resistance between any two terminals in one combination equals the net resistance between the corresponding terminals in the other combination.

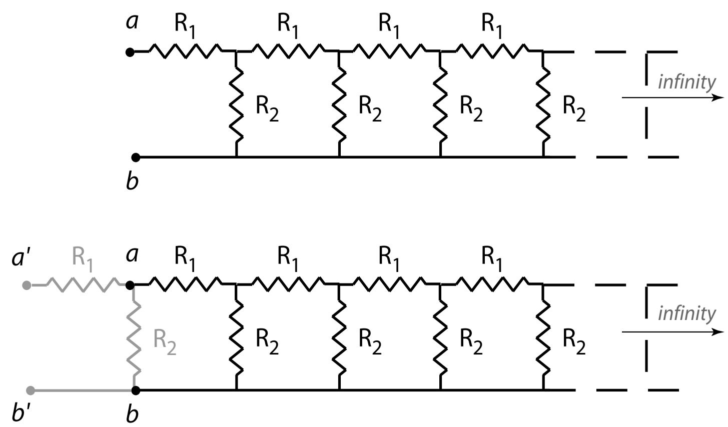

Figure 7: Top: The circuit represents an infinite ladder of resistors. The goal is to find the net resistance between terminals a and b. Bottom: If the ladder is indeed infinite adding another section to the ladder should not change the net resistance between terminals a and b.

Kirchoff's laws

In a circuit involving various combination of resistors with various branches we can apply the rules alluded to earlier. Consider the circuit of Figure 8. We assign a current for each of the resistors including a specific direction for the current arrow. Now we apply the equivalent of energy conservation and charge conservation - and this leads to Kirchoff's laws:

1. For any complete circuit the sum of all voltage changes across each element is zero.
2. At any node the sum of the currents flowing in equals the sum of currents flowing out.

As an example of the application of the first of the two laws:

$$V - i_1 R_1 = 0 \tag{13}$$

$$V - i_3 R_3 - i_4 R_4 = 0 \tag{14}$$

$$V - i_6 R_6 = 0 \tag{15}$$

and an example of the application of the second law:

$$i_2 + i_3 = i_4 + i_5 \tag{16}$$

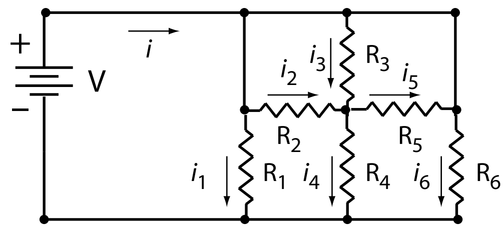

Figure 8: A multi-branch circuit with combinations of resistors.

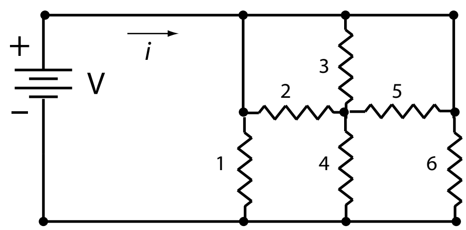

Figure 9: A multi-branch circuit with combinations of resistors.

In this particular example we assume the battery voltage and all resistors are specified. Then there are seven unknown currents - one through each of the resistors and the battery. By applying Kirchoff's laws you can find seven equations to solve for the seven unknowns.

If it happens that the this procedure yields a negative value for a current, then the assumption about the direction of the current was wrong.

This is the brute force technique for solving d.c. circuits but application of the rules for combining series and parallel combinations of resistors and looking for voltage and current dividers might provide more insight. In Figure 9 we re-draw the circuit of Figure 8 removing the current arrows. In one of the problems you are asked to find ways to simplify the circuit by looking for series and parallel combinations of resistors and then treating parts of the circuit as current or voltage dividers to determine the unknown currents.

Practical considerations

Ideal batteries and wires

When we draw an electrical circuit it is assumed that the wires (represented by lines) connecting circuit elements are ideal – *i.e.* have no resistance. This is of course an approximation since any real wire has some small resistance but we can often ignore it. The battery is also assumed to be ideal. An ideal battery supplies the same voltage independent of current, but this is also an approximation. Any real battery has some small internal resistance so that the real battery delivers a voltage that depends on current.

In Figure 10 a real battery is shown as a series combination of an ideal battery providing voltage V in series with the battery's internal resistance r. So the voltage that actually appears across the

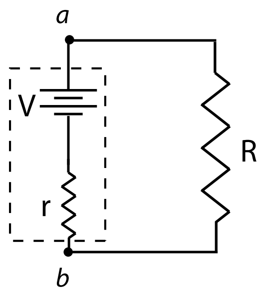

Figure 10: A real battery can be approximated by an ideal battery in series with a internal resistance r.

terminals of the battery (a and b) is $V - ir$ where $i = V/(R + r)$.

Resistor color codes

Resistors are usually labeled with a color code of four bands which give the value of the resistor and tolerance as shown in Figure 11. The digits associated with the first two bands yields a two-digit number that is then followed by a number of zeros specified by the color of the third band. Numbers (0 through 9) are represented by colors in the order black, brown, red, orange, yellow, green, blue, violet, gray and white. The fourth band is the tolerance: gold is 5% and silver is 10%.

For example, a resistor with color bands red-red-orange-gold has a value of 22000 Ω with a tolerance of $\pm 5\%$. Often one refers to a 1000 Ω resistor as a 1 $k\Omega$ or simply $1k$ resistor.

Other resistor parameters

Resistors are also rated for maximum voltage and maximum wattage (power rating). The power rating is determined by the physical size. For example typical $\frac{1}{4}$ W resistors are cylinders about $\frac{1}{4}$ in long and $\frac{1}{16}$ in in diameter while $\frac{1}{2}$ W resistors are cylinders about $\frac{3}{8}$ in long and $\frac{1}{8}$ in in diameter.

The resistance of a resistor also increases with temperature.

Resistance of wires

The resistance of wires depends on the metal used for the wires, the length of the wire (ℓ) and its cross-sectional area (A) in the following way:

$$R = \rho \frac{\ell}{A} \tag{17}$$

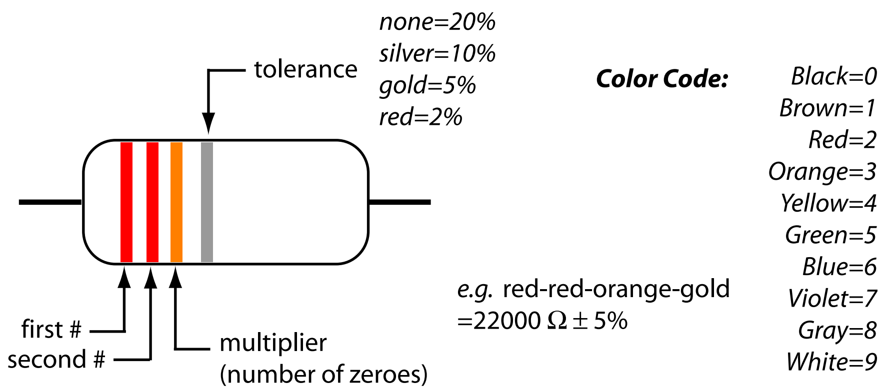

Figure 11: Color code for resistors

where ρ is the resistivity and depends on the material. For example, copper has a resistivity of $1.68\ \times 10^{-8}\ \Omega \cdot m$. Silver and aluminum have resistivities that are 95% and 160% that of copper respectively. Not surprisingly the wire resistivity increases with increasing length and decreasing cross-sectional area.

Wire is often specified in AWG gauges where a higher gauge corresponds to a smaller wire diameter. For 1 km of copper wire, 12 gauge wire (typical household wiring) has a resistance of 5 Ω and 24-gauge wire (telephone wire) has a resistance of 85 Ω.

Measuring current, voltage and resistance

Digital multimeters can measure current and voltage (both d.c. and a.c.) and resistance. You will be using multimeters in several of the labs.

Problems

Problem 1

Suppose you have an application that requires a $1k$ resistance with at least a $\frac{1}{2}$ W rating but all you have are many $1k$ $\frac{1}{4}$ W 10% resistors. How many resistors would you need to satisfy your requirement and how would you arrange them? What would the maximum power rating be for the net combination and what would the tolerance be?

Problem 2

For the circuit of Figure 5 what value of R_1 (in terms of R_0) will result in a net resistance of R_0 between terminals a and b?

Problem 3

For the circuit of Figure 6 explicitly show that the resistance between any two terminals of combination (1) is equal to the resistance between the corresponding terminals of combination (2).

Problem 4

For the circuit of Figure 8 write down a set of seven linear equations using Kirchoff's laws to solve for the seven unknown currents.

Problem 5

For the circuit of Figure 9 assume the all resistors are identical and have resistance R. Use your answer to problem 4 to find the currents in each resistor and the battery.

Re-examine the circuit carefully and use the rules for combining parallel and series resistors to find the net resistance across the battery.

Problem 6

For the circuit of Figure 10 assume that the resistance R is variable and the internal resistance r of the battery is fixed. For what value of R will the power dissipated by R be a maximum?

Problem 7

Consider the left circuit of Figure 3. Current I splits into I_1 and I_2 where $I = I_1 + I_2$. Now find the relation between I_1 and I and I_2 and I by requiring only that the total power delivered to the two resistors be a minimum. *Do not use the rule for parallel resistors.* Compare your answer with equations 11 and 12 for the current divider.

Problem 8

Suppose you hook up two batteries of voltages V_1 and V_2 in series. What are the possible voltages across the combination?

Suppose you hook up two batteries in parallel? What are the possible outcomes?

Problem 9

For the circuit of Figure 7 find the net resistance between terminals a and b in terms of R_1 and R_2. Use the fact that the ladder of resistors is infinite so that adding another section should not change the net resistance (compare the top and bottom circuits of Figure 7).

Suppose you were to apply a battery of voltage V between terminals a and b. What would the voltage be across the leftmost R_2 resistor in the top circuit of Figure 7?

What would the voltage be across the next leftmost R_2 resistor in the top circuit of Figure 7?

Suppose we want the voltage across the leftmost R_2 resistor to be $V/2$ and across the next R_2 resistor to be $V/4$ and so on. What should the be the relation between R_2 and R_1?

Of course we cannot truly have an infinite ladder – so how should the right end of the network be terminated so as not to change anything to the left (that is compared to the infinite ladder case)?